NEW TYPE OF SOLUTIONS TO A SLIGHTLY SUBCRITICAL HÉNON TYPE PROBLEM ON GENERAL DOMAINS

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Abstract. We consider the following slightly subcritical problem

$$(\wp_\varepsilon) \qquad \begin{cases} -\Delta u = \beta(x)|u|^{p-1-\varepsilon}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $3 \le n \le 6$, $p := \frac{n+2}{n-2}$ is the Sobolev critical exponent, ε is a small positive parameter and $\beta \in C^2(\overline{\Omega})$ is a positive function. We assume that there exists a non degenerate critical point $\xi_* \in \partial\Omega$ of the restriction of β to the boundary $\partial\Omega$ such that

$$\nabla(\beta(\xi_*)^{\frac{-2}{p-1}}) \cdot \eta(\xi_*) > 0,$$

where η denotes the inner normal unit vector on $\partial\Omega$. Given any integer $k \ge 1$, we show that for $\varepsilon > 0$ small enough problem $(\wp_\varepsilon)$ has a positive solution, which is a sum of k bubbles which accumulate at ξ_* as ε tends to zero. We also prove the existence of a sign changing solution whose shape resembles a sum of a positive bubble and a negative bubble near the point ξ_*.

Keywords: Hénon Problem, Critical exponent, Blowing up solutions.

AMS subject classification: 35J60, 35J20, 35J25.

1. Introduction and statement of main results

We consider the non autonomous almost critical problem

$$(\wp_\varepsilon) \qquad \begin{cases} -\Delta u = \beta(x)|u|^{p-1-\varepsilon}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $3 \le n \le 6$, $p := \frac{n+2}{n-2}$ is the Sobolev critical exponent, ε is a small positive parameter and the function $\beta \in C^2(\overline{\Omega})$ is positive.

Since problem $(\wp_\varepsilon)$ is subcritical, standard variational methods yields the existence of an infinite number of sign changing solutions and at least one positive solution, see [2]. Unfortunately, the variational approach gives very little information about the behaviour of these solutions.

A special case of problem $(\wp_\varepsilon)$ is the following

$$(\wp^1_\varepsilon) \qquad \begin{cases} -\Delta u = |u|^{p-1-\varepsilon}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

This problem has been extensively studied in the last decades and many works has been devoted to study existence and asymptotic behaviour of solutions. We refer to the pioneering work by Bahri-Li-Rey in [3], where they proved that positive solutions to problem $(\wp^1_\varepsilon)$ either converge to a positive solution of the critical problem $(\wp^1_0)$ or blow up at a finite number of points in Ω

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as ε goes to zero. More precisely, if (u_ε) is a bounded sequence in $H_0^1(\Omega)$ of positive solutions to $(\wp_\varepsilon^1)$, then (up to a subsequence) we have

$$u_\varepsilon = u_0 + \sum_{i=1}^{k} \alpha_i^\varepsilon PU_{\lambda_i^\varepsilon, \xi_i^\varepsilon} + v_\varepsilon$$

where u_0 is a nonnegative solution to $(\wp_0^1)$, $k \in \mathbb{N}$, v_ε goes to zero in $H_0^1(\Omega)$. Here either $k = 0$ or $u_0 = 0$ and the function $PU_{\delta,\xi}$ is the orthogonal projection onto $H_0^1(\Omega)$ of the "bubble" given by

$$U_{\delta,\xi}(x) := (n(n-2))^{\frac{n-2}{4}} \frac{\lambda^{\frac{n-2}{2}}}{(\delta^2 + |x-\xi|^2)^{\frac{n-2}{2}}},$$

with $\delta > 0$ and $\xi \in \mathbb{R}^n$. This family of functions represent all solutions to

$$-\Delta u = u^{\frac{n+2}{n-2}} \quad \text{in } \mathbb{R}^n, \quad u \in D^{1,2}(\mathbb{R}^n), \quad u > 0.$$

In the case where a positive solution concentrates at a single point, it was proved in [6, 13, 18] that this concentration point must be a critical point of the Robin function:

$$x \to H(x,x),$$

where $H(x,y)$ stands for the regular part of the Green function for the Laplacian in Ω with Dirichlet boundary condition (see (5.1)). Results about multiplicity of positive solutions for problem $(\wp_\varepsilon^1)$ with multiple blow up points have been also obtained, see [19] for instance.

The presence of the potential β in $(\wp_\varepsilon)$ plays a crucial role for existence of positive solutions with a large number of blow-up points. Indeed, it has been shown in [3] that problem $(\wp_\varepsilon^1)$ does not admit positive solutions which concentrates at k points as ε goes to zero if k is large enough. However, for problem $(\wp_\varepsilon)$, Pistoia and Serra studied in [16] the particular case where $\beta(x) = |x|^\alpha$, $\alpha > 0$, and Ω is the unit ball B_1, namely they considered the problem

$$\begin{cases} -\Delta u = |x|^\alpha |u|^{p-1-\varepsilon} u & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1. \end{cases}$$

They showed that, if ε is small enough, then the above problem has a positive solution which concentrates and blow-up at ℓ points at the boundary ∂B_1. Moreover, the solutions constructed in [16] are invariant under the group of linear symmetries $G_1 \times \mathrm{O}(n-2)$, where $G_1 \subset \mathrm{O}(2)$ is the group generated by the rotations of angle $\frac{2\pi}{\ell}$. See also Peng [15] who constructed similar solutions with more general symmetries. We also refer the reader to the papers [7, 8] and some references therein, where asymptotic behaviour of the ground state solution (as ε tends to zero) has been considered. Precisely, it has been proven that the ground state concentrates at a single point which approaches the boundary when ε tends to zero.

In this paper, we prove the existence of new type of concentrating positive solutions to problem $(\wp_\varepsilon)$. Precisely, we show, under some suitable conditions on the function β, that $(\wp_\varepsilon)$ has a positive solutions whose asymptotic profile is a sum of k bubbles that concentrates and blow up at *a single point* at the boundary. It is worthmentioning that for $k = 1$, our results work for any dimension $n \geq 3$ and this can be seen in particular as a generalisation the main results in [15, 16], to the problem $(\wp_\varepsilon)$ for general domains, see Corollary 1.1 below. For $k \geq 2$, our results here are valid for dimensions $3 \leq n \leq 6$. This restriction on the dimension is technical and we believe that with more accurate analysis they can be generalised to any dimension $n \geq 3$.

For $x \in \partial\Omega$ let $\eta(x)$ be the unitary inner normal vector to $\partial\Omega$ at x. The following condition on β will be assumed throughout the paper: there exists a non degenerate critical point $\xi_* \in \partial\Omega$ of the restriction of β to the boundary $\partial\Omega$ such that

$$\nabla(\beta(\xi_*)^{-\frac{2}{p-1}}) \cdot \eta(\xi_*) > 0.$$

This assumption can be rewritten as

$$\text{there is } \lambda > 0 \text{ such that } \nabla(\beta(\xi_*)^{-\frac{2}{p-1}}) = \lambda \eta(\xi_*). \tag{1.1}$$

We will first prove the following multiplicity result.

Theorem 1.1. *Assume that condition (1.1) holds true, that that* $D^2(\beta^{-\frac{2}{p-1}}|_{\partial\Omega})(\xi_*)$ *is negative definite and* $3 \le n \le 6$. *Then, for every* $k \in \mathbb{N}$ *there exists* ε_0 *positive such that if* $\varepsilon \in (0, \varepsilon_0)$ *problem* $(\wp_\varepsilon)$ *has a positive solution* u_ε *of the form*

$$u_\varepsilon = \sum_i^k \beta(\xi_{\varepsilon,i})^{-\frac{1}{p-1}} U_{\delta_{\varepsilon,i},\xi_{\varepsilon,i}} + o(1) \quad \text{in } D^{1,2}(\Omega),$$

where

$$\xi_{\varepsilon,i} = \xi^0_{\varepsilon,i} + \tau_{\varepsilon,i}\eta(\xi^0_{\varepsilon,i}), \quad \text{with } \xi^0_{\varepsilon,i} := \xi_* + \varepsilon^{\frac{n+1}{n+2}} v_i + o(\varepsilon^{\frac{n+1}{n+2}} v_i), \qquad v_i \in T_{\xi_*}\partial\Omega$$

and (up to a subsequence)

$$\varepsilon^{-\frac{n-1}{n-2}}\delta_{\varepsilon,i} \to d_i > 0, \quad \varepsilon^{-1}\tau_{\varepsilon,i} \to t_i > 0 \text{ and } \varepsilon^{-\frac{n+1}{n+2}}|\xi_{\varepsilon,i} - \xi_{\varepsilon,j}| \to |v_i - v_j| > 0,$$

for all $i, j = 1, \dots, k$.

As a consequence of the above theorem, we have

Corollary 1.1. *Assume* $3 \le n \le 6$. *Suppose that* $\xi^1_*, \dots, \xi^\ell_*$ *are non degenerate critical points of the restriction of* β *to the boundary* $\partial\Omega$ *such that*

$$\nabla(\beta(\xi^j_*)^{-\frac{1}{p-1}}) \cdot \eta(\xi^j_*) > 0, \quad \text{and} \quad D^2(\beta^{-\frac{2}{p-1}}|_{\partial\Omega})(\xi^j_*) \quad \text{are negative definite } \forall\, j = 1, \dots, \ell.$$

Then, for ε *sufficiently small, problem* $(\wp_\varepsilon)$ *has a positive solution* u_ε *of the form*

$$u_\varepsilon = \sum_{j=1}^{\ell}\sum_{i=1}^{k} \beta(\xi^j_{\varepsilon,i})^{-\frac{1}{p-1}} U_{\delta_{\varepsilon,i},\xi^j_{\varepsilon,i}} + o(1) \quad \text{in } D^{1,2}(\Omega),$$

where $\xi^j_{\varepsilon,i} \to \xi^j_*$ *(up to a subsequence) for each* $i = 1, \dots, k$ *and* $j = 1, \dots, \ell$.

Remark 1.2. *In the previous corollary, if we restrict* $k = 1$ *(simple concentration at each* ξ^j_**) the result is true for any* $n \ge 3$, *see Subsection 3.2 below for details.*

Although we stated the result requiring β *positive, it is necessary to assume it is positive near the concentration points.*

The phenomenon of multiple concentration near a point of the boundary found in Theorem 1.1 is similar to the multiplicity result of Wei and Yan [20] for a critical Lazer-McKenna conjecture in dimensions $n \ge 6$, and to the paper by del Pino, Musso and Pistoia [12] where bubble tower solutions to a Neumann Lin-Ni-Takagi problem has been constructed in both slightly subcritical and slightly supercritical regimes.

The second purpose of this paper is to study existence and properties of sing-changing solutions for problem $(\wp_\varepsilon)$. If we consider the problem $(\wp^1_\varepsilon)$, then multiple peak nodal solutions always exits. Indeed Bartsch, Micheletti and Pistoia [5] built a solution with exactly one positive and one negative concentration point. In addition, under symmetry assumptions on Ω, a solution to $(\wp^1_\varepsilon)$ with exactly two positive and two negative blow-up points was constructed in [4].

Bubble-tower solutions to problem $(\wp^1_\varepsilon)$ have been constructed in [14, 17]. The shape of these solutions is a superpositions of positive bubbles and negative bubbles blowing up at a single point with different velocities.

Our second result show that condition (1.1) guarantees the existence of a solution to problem $(\wp_\varepsilon)$ with one positive and one negative concentration points, which blow up at a the single point ξ_* at the boundary. More precisely we have the next result.

Theorem 1.3. *Assume that condition (1.1) holds true, that* $D^2(\beta^{-\frac{2}{p-1}}|_{\partial\Omega})(\xi_*)$ *is positive definite and* $3 \le n \le 6$. *Then, there exists* $\varepsilon_0 > 0$ *such that if* $\varepsilon \in (0, \varepsilon_0)$ *then problem* $(\wp_\varepsilon)$ *has a sign changing solution* u_ε *of the form*

$$u_\varepsilon = \beta(\xi_{\varepsilon,1})^{-\frac{1}{p-1}} U_{\delta_{\varepsilon,1},\xi_{\varepsilon,1}} - \beta(\xi_{\varepsilon,2})^{-\frac{1}{p-1}} U_{\delta_{\varepsilon,2},\xi_{\varepsilon,2}} + o(1) \quad \text{in } D^{1,2}(\Omega)$$

where

$$\xi_{\varepsilon,i} = \xi^0_{\varepsilon,i} + \tau_{\varepsilon,i}\eta(\xi^0_{\varepsilon,i}), \quad \xi^0_{\varepsilon,i} \in \partial\Omega, \qquad \xi^0_{\varepsilon,i} \to \xi_*,$$

and (up to a subsequence)

$$\varepsilon^{-\frac{n-1}{n-2}}\delta_{\varepsilon,i} \to d_i > 0, \quad \varepsilon^{-1}\tau_i \to t_i > 0, \quad \varepsilon^{-\frac{n+1}{n+2}}|\xi_{\varepsilon,1} - \xi_{\varepsilon,2}| \to |v_1 - v_2| > 0$$

for $i = 1, 2$.

Theorems 1.1 and 1.3 are related to a paper by Ackermann, Clapp and Pistoia, see [1]. They studied a supercritical problem which can be reduced, using rotational symmetries, to a problem similar to $(\wp_\varepsilon)$ given by

$$\begin{cases} -\mathrm{div}(\beta(x)\nabla u) = \beta(x)|u|^{p-1-\varepsilon}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

They proved, under a similar condition to (1.1), that problem (1.2) has a solution which has the shape of one bubble and concentrate and blow up at one point at the boundary.

The arguments used in this paper for the problem $(\wp_\varepsilon)$ can be adapted to prove the analogous results for problem (1.2). This leads to the construction of new types of solutions for some supercritical problems.

Some results here are valid for dimensions $3 \le n \le 6$. This is related to the fact that we need the size of the error term to be controlled in some appropriate norms by $o(\varepsilon^{\frac{2(n+1)}{(n+2)}})$ which follows by Lemma 2.1 for $3 \le n \le 6$. We believe that our result can be extended to higher dimensions by adding further improvement to the approximate solution constructed in Section 2.1.

The paper is organized as follows: We first recall in Section 2 some preliminary results. Section 3 will be mainly devoted to the proofs of our main results. In these proofs we will need some asymptotic expansions of the reduced energy functional, which is developed in Section 4. Finally, in Section 5 we give some boundary estimates for the Green's function.

2. Preliminaries

Let us first introduce the function

$$U_{\delta,\xi}(x) := a_0 \frac{\delta^{\frac{n-2}{2}}}{(\delta^2 + |x-\xi|^2)^{\frac{n-2}{2}}}, \quad a_0 := (n(n-2))^{\frac{n-2}{4}}, \quad \delta > 0$$

which corresponds up to translations and dilations to the standard bubble, namely, the unique positive solution to the problem

$$\begin{cases} -\Delta U = U^p & \text{in } \mathbb{R}^n, \\ U \in D^{1,2}(\mathbb{R}^n), \end{cases}$$

where $n \ge 3$ and $p = \frac{n+2}{n-2}$. We next define the function

$$W_{\delta,\xi} := \beta(\xi)^{-\frac{1}{p-1}} U_{\delta,\xi}.$$

It is easy to see that $W_{\delta,\xi}$ is a solution of the equation

$$-\Delta W(x) = \beta(\xi) W^p(x) \quad \text{in } \mathbb{R}^n.$$

Let us consider the orthogonal projection

$$P : D^{1,2}(\mathbb{R}^n) \to H^1_0(\Omega)$$

defined by : given $W \in D^{1,2}(\mathbb{R}^n)$, we let PW to be defined as the unique solution to the problem

$$-\Delta\left(PW\right) = -\Delta W \quad \text{in } \Omega, \qquad PW = 0 \quad \text{on } \partial\Omega.$$

Next we describe the solutions that we are looking for with multiple concentration on a single point on the boundary ($k \geq 2$). For simple concentration (k=1) see Subsection 3.2 for more details. In Theorems 1.1 and 1.3 we found solutions of the form

$$u_\varepsilon = \sum_i^k (-1)^{\lambda_i} \beta(\xi_{i,\varepsilon})^{-\frac{1}{p-1}} PU_{\delta_{i,\varepsilon},\xi_{i,\varepsilon}} + \phi, \tag{2.1}$$

for fixed $\lambda_i \in \{0,1\}$. For $i = 1,\dots,k$, the dilation parameters $\delta_{i,\varepsilon}$ will be chosen of the form

$$\delta_{i,\varepsilon} = \varepsilon^{\frac{n-1}{n-2}} d_i \text{ for some } d_i > 0, \tag{2.2}$$

and the concentration points satisfy

$$\xi_{i,\varepsilon} = \xi^0_{i,\varepsilon} + \tau_{i,\varepsilon}\eta(\xi^0_{i,\varepsilon}), \qquad \xi^0_{i,\varepsilon} \in \partial\Omega,$$

where $\tau_{i,\varepsilon} = \varepsilon t_i$ for some $t_i > 0$ and $\xi^0_{i,\varepsilon}$ is given by

$$\xi^0_{i,\varepsilon} = \xi_* + \rho v_i + \eta(\xi_*) g(\rho v_i), \tag{2.3}$$

with

$$v_i \in T_{\xi_*}\partial\Omega := \Big\{ v \in \mathbb{R}^n : \eta(\xi_*)\cdot v = 0 \Big\}, \tag{2.4}$$

with $\rho = \varepsilon^{\frac{n+1}{n+2}}$ and $g : T_{\xi_*}\partial\Omega \to \mathbb{R}$ is a function which satisfies

$$g(0) = 0 \text{ and } \nabla g(0) = 0.$$

Here $T_{\xi_*}\partial\Omega$ stands for the tangent space of $\partial\Omega$ at the point ξ^*.

The function ϕ in (2.1) is small in a sense to be determined later and is to be found using a classical fixed point argument.

We will next introduce the configuration space where the dilation parameters and the concentration points lie. We set $\mathbf{d} = (d_1,\dots,d_k)$, $\mathbf{t} = (t_1,\dots,t_k)$ and $\mathbf{v} = (v_1,\dots,v_k)$, then the configuration space is given by

$$\Lambda := \Big\{ (\mathbf{d},\mathbf{t},\mathbf{v}) \in (0,\infty)^k \times (0,\infty)^k \times (T_{\xi_*}\partial\Omega)^k : v_i \neq v_j \text{ for } i,j = 1,\dots,k, i \neq j \Big\}.$$

For simplicity we will write

$$V^\varepsilon_{\mathbf{d},\mathbf{t},\mathbf{v}} := \sum_i (-1)^{\lambda_i} b_i PU_i, \tag{2.5}$$

where we have set $b_i := \beta(\xi_i)^{-\frac{1}{p-1}}$ and $U_i := U_{\delta_i,\xi_i}$.

2.1. **Lyapunov-Schmidt reduction procedure.** In this subsection we will recall the main ideas about the Lyapunov-Schmidt reduction procedure which is a crucial step to find solutions of the form (2.1).

The first step to construct solutions to problem $(\wp_\varepsilon)$, we need to solve some auxiliary problem. Given $(\mathbf{d},\mathbf{t},\mathbf{v}) \in \Lambda$, we consider the spaces

$$\mathcal{K}^\varepsilon_{\mathbf{d},\mathbf{t},\mathbf{v}} = \text{span}\Big\{ P\left(\frac{\partial U_{\delta_i,\xi_i}}{\partial \xi^j_i}\right), P\left(\frac{\partial U_{\delta_i,\xi_i}}{\partial \delta_i}\right) : i = 1,\dots,k,\ j = 1,\dots,n \Big\}$$

$$\mathcal{K}^{\varepsilon,\perp}_{\mathbf{d},\mathbf{t},\mathbf{v}} = \Big\{ \phi \in H^1_0(\Omega) : \int_\Omega \nabla\phi\cdot\nabla\psi = 0 \ \ \forall\, \psi \in \mathcal{K}^\varepsilon_{\mathbf{d},\mathbf{t},\mathbf{v}} \Big\}.$$

The following result hold.

Lemma 2.1. *Assume that for some $\tau > 0$ and a fixed constant $\bar{C}$, $\tau/\bar{C} \leq \tau_{i,\varepsilon} \leq \bar{C}\tau$. Then there exist $\varepsilon_0 > 0$ and a constant $C > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ and all $(\boldsymbol{d,t,v}) \in \Lambda$ there exists a unique $\phi^\varepsilon_{\boldsymbol{d,t,v}} \in \mathcal{K}^{\varepsilon,\perp}_{\boldsymbol{d,t,v}}$ which satisfies*

$$\Delta(V^\varepsilon_{\mathbf{d},\mathbf{t},\mathbf{v}} + \phi^\varepsilon_{\boldsymbol{d,t,v}}) + \beta(x)|V^\varepsilon_{\mathbf{d},\mathbf{t},\mathbf{v}} + \phi^\varepsilon_{\boldsymbol{d,t,v}}|^{p-1-\varepsilon}\left(V^\varepsilon_{\mathbf{d},\mathbf{t},\mathbf{v}} + \phi^\varepsilon_{\boldsymbol{d,t,v}}\right) \in \mathcal{K}^\varepsilon_{\boldsymbol{d,t,v}} \tag{2.6}$$

and

(2.7)
$$\|\phi^{\varepsilon}_{\boldsymbol{d,t,v}}\| := \left(\int_\Omega |\nabla \phi^{\varepsilon}_{\boldsymbol{d,t,v}}|^2\right)^{\frac{1}{2}} = \begin{cases} \mathcal{O}\left(\varepsilon|\log\varepsilon| + \tau + (\frac{\delta}{\tau})^{\frac{n+2}{2}}\right) & for \qquad n \geq 7 \\ \mathcal{O}\left(\varepsilon|\log\varepsilon| + \tau + (\frac{\delta}{\tau})^{4}\log^{\frac{2}{3}}(\frac{\delta}{\tau})\right) & for \qquad n = 6 \\ \mathcal{O}\left(\varepsilon|\log\varepsilon| + \tau + (\frac{\delta}{\tau})^{n-2}\right) & for \quad n = 3,\ 4 \ \text{and}\ 5, \end{cases}$$

where $\delta = \max \delta_i$. *Moreover the map* $\Lambda \to H^1_0(\Omega)$, *defined by* $(\mathbf{d},\mathbf{t},\mathbf{v}) \mapsto \phi^{\varepsilon}_{(\boldsymbol{d,t,v})}$ *is of class* C^1 *and*

$$\|\nabla_{(\boldsymbol{d,t})}\, \phi^{\varepsilon}_{\boldsymbol{d,t,v}}\| = O(\|\phi^{\varepsilon}_{\boldsymbol{d,t,v}}\|). \tag{2.8}$$

Proof. The proof of existence and estimate (2.7) can be found in [16], see Proposition 2 there. To prove estimate (2.8), we use the fact that the solution $\phi^{\varepsilon}_{\mathbf{d,t,v}}$ of (2.6) is found by a fixed point argument. It satisfies an equation of the form

$$\phi^{\varepsilon}_{\mathbf{d,t,v}} = A^{\varepsilon}_{\mathbf{d,t,v}}(E + N(\phi^{\varepsilon}_{\mathbf{d,t,v}})) \tag{2.9}$$

where $A^{\varepsilon}_{\mathbf{d,t,v}} : \mathcal{K}^{\varepsilon,\perp}_{\mathbf{d,t,v}} \to \mathcal{K}^{\varepsilon,\perp}_{\mathbf{d,t,v}}$ is given by

$$A^{\varepsilon}_{\mathbf{d,t,v}}(\cdot) = \Pi^{\perp}_{\mathbf{d,t,v}}\, i^*(\cdot),$$
$$E = -\Delta V^{\varepsilon}_{\mathbf{d,t,v}} - \beta |V^{\varepsilon}_{\mathbf{d,t,v}}|^{p-1-\varepsilon}(V^{\varepsilon}_{\mathbf{d,t,v}})$$

and

$$N(\phi) = \beta\Big(|V^{\varepsilon}_{\mathbf{d,t,v}} + \phi|^{p-1-\varepsilon}(V^{\varepsilon}_{\mathbf{d,t,v}} + \phi) - |V^{\varepsilon}_{\mathbf{d,t,v}}|^{p-1-\varepsilon}(V^{\varepsilon}_{\mathbf{d,t,v}}) - p|V^{\varepsilon}_{\mathbf{d,t,v}} + \phi|^{p-1-\varepsilon}\phi\Big)$$

In the above $\Pi^{\perp}_{\mathbf{d,t,v}}$ is the orthogonal projection on $\mathcal{K}_{\mathbf{d,t,v}}$ and $i^* : L^{\frac{2n}{n-2}}(\Omega) \to H^1_0(\Omega)$ is the adjunct of the standard immersion operator $i : H^1_0(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$. Differentiating (2.9) with respect to the parameters $\mathbf{t}$, we formally get

$$\partial_{\mathbf{t}}\phi^{\varepsilon}_{\mathbf{d,t,v}} = (\partial_{\mathbf{t}} A^{\varepsilon}_{\mathbf{d,t,v}})(E + N(\phi^{\varepsilon}_{\mathbf{d,t,v}})) + A^{\varepsilon}_{\mathbf{d,t,v}}\Big(\partial_{\mathbf{t}} E + \partial_{\mathbf{t}} N(\phi^{\varepsilon}_{\mathbf{d,t,v}})\Big).$$

There is a similar formula for the derivative with respect to $\mathbf{d}$. Reasoning as in the proof of Lemma 4.1 in [11] and using similar arguments as in the proof of estimate (2.7), one gets

$$\|\nabla_{\mathbf{d,t}}\phi^{\varepsilon}_{\mathbf{d,t,v}}\| = O(\|\phi^{\varepsilon}_{\mathbf{d,t,v}}\|). \tag{2.10}$$

□

Now, let $J_\varepsilon : H^1_0(\Omega) \to \mathbb{R}$ be the energy functional associated to problem $(\wp_\varepsilon)$:

$$J_\varepsilon(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p+1-\varepsilon}\int_\Omega \beta(x)|u|^{p+1-\varepsilon}$$

Solutions to problem $(\wp_\varepsilon)$ can be found as critical points of the functional J_ε. We introduce the reduced energy functional $\mathcal{I}_\varepsilon : \Lambda \longrightarrow \mathbb{R}$ defined by

$$\mathcal{I}_\varepsilon(\mathbf{d,t,v}) = J_\varepsilon(V^{\varepsilon}_{\mathbf{d,t,v}} + \phi^{\varepsilon}_{\mathbf{d,t,v}}). \tag{2.11}$$

The next lemma, which is a consequence of Lemma 2.1, reduces the existence of solutions to problem $(\wp_\varepsilon)$ to the one of finding critical points of the reduced energy functional $\mathcal{I}_\varepsilon$.

Lemma 2.2. *The element* $(\boldsymbol{d,t,v}) \in \Lambda$ *is a critical point of* $\mathcal{I}_\varepsilon$ *if and only if the function* $u_\varepsilon = V^{\varepsilon}_{\mathbf{d,t,v}} + \phi^{\varepsilon}_{\mathbf{d,t,v}}$ *is a critical point of the functional* J_ε.

Proof. The proof is similar to the one of Proposition 1 in [3]. We omit it here. □

3. Proof of the main theorems.

In this section we prove our main results. We will treat the case of multiple concentration on a single point on the boundary, namely when $k \geq 2$. For the case of simple concentration (k=1) the proof is easier and is done in Subsection 3.2. Using Lemma 2.2, the proof of Theorems 1.1 and 1.3 is then reduced to finding critical points of the reduced functional $\mathcal{I}_\varepsilon$ defined in (2.11). To do so we will need the asymptotic expansion of the reduced energy whose proof is given in the next section. Let $r = -\frac{2}{p-1}$ and define the function

$$\Gamma(\xi_*, v) := D^2(\beta^r|_{\partial\Omega})(\xi_*)[v, v]. \tag{3.1}$$

Since the function g satisfies

$$g(0) = 0 \text{ and } \nabla g(0) = 0$$

it is easy to see that

$$\Gamma(\xi_*, v) = \nabla(\beta(\xi_*)^r) \cdot \eta(\xi_*)(D^2 g(\xi_*)v \cdot v) + D^2(\beta(\xi_*)^r)v \cdot v.$$

The next lemma will be proved in Section 4.

Lemma 3.1. *The functional $\mathcal{I}_\varepsilon : \Lambda \to \mathbb{R}$ has the following asymptotic expansion*

$$\mathcal{I}_\varepsilon(\mathbf{d}, \mathbf{t}, \mathbf{v}) = c_1\beta(\xi_*)^r + \varepsilon\psi_0(\xi_*) + \varepsilon\psi_1(\mathbf{d}, \mathbf{t}) + \rho^2\psi_2(\mathbf{d}, \mathbf{t}, \mathbf{v}) + \psi_3(\mathbf{d}, \mathbf{t}, \mathbf{v}) + o(\epsilon^2) \tag{3.2}$$

C^1-uniformly on compact sets of Λ. Here c_i are positive constants and the function ψ_j's are respectively given by

$$\begin{aligned}\psi_0(\xi_*) &:= \frac{1}{p+1}\left(k\gamma_1\beta(\xi_*)^r \log(\beta(\xi_*^0)^{\frac{r}{2}}) - k\gamma_1\beta(\xi_*)^r \frac{n-1}{2}\log(\varepsilon) + k\beta(\xi_*)^r \int_{\mathbb{R}^n} U_{1,0}\log(U_{1,0})\right) \\ &- \frac{\gamma_1}{(p+1)^2} k\beta(\xi_*)^r,\end{aligned}$$

$$\psi_1(\mathbf{d}, \mathbf{t}) = \sum_j c_2\left(\frac{d_j}{t_j}\right)^{n-2} + c_3\nabla(\beta(\xi_*)^r) \cdot \eta(\xi_*)t_j - c_4\log(d_j) + O(\varepsilon^{\frac{2}{n-2}})\mathcal{R}(\mathbf{d}, \mathbf{t}), \tag{3.3}$$

where $\mathcal{R}$ is a bounded smooth function on its arguments which does not depend on the variables $(v_1, \ldots, v_k)$,

$$\psi_2(\mathbf{d}, \mathbf{t}, \mathbf{v}) = -c_5\sum_{i>j}(-1)^{\lambda_i}(-1)^{\lambda_j}\beta(\xi_*)^r \frac{d_i^{\frac{n-2}{2}} d_j^{\frac{n-2}{2}} t_i t_j}{|v_i - v_j|^n} + \sum_j \Gamma(\xi_*, v_j) \tag{3.4}$$

and where ψ_3 satisfies

$$\psi_3(\mathbf{d}, \mathbf{t}, \mathbf{v}) = \begin{cases} \mathcal{O}\left(\epsilon^2 |\log\varepsilon|^{\frac{4}{3}}\right) & \text{for} \qquad n = 6 \\ \\ \mathcal{O}\left(\varepsilon^2 |\log\varepsilon|^2\right) & \text{for} \quad n = 3,\ 4 \ \text{ and } \ 5. \end{cases}$$

Moreover for $\mathbf{d}, \mathbf{t}, \mathbf{v} \in \Lambda$

$$|\nabla_{(\mathbf{d},\mathbf{t})}\psi_i(\mathbf{d}, \mathbf{t}, \mathbf{v})| = O(\psi_i(\mathbf{d}, \mathbf{t}, \mathbf{v})) \qquad i = 1, 2, 3. \tag{3.5}$$

3.1. **Proof of Theorems 1.1 and 1.3.** To prove our main results Theorems 1.1 and 1.3, we need to show that the functional $\mathcal{I}_\varepsilon$ has a critical point. Using that $\nabla(\beta(\xi_*)^r) \cdot \eta(\xi_*) > 0$, see (1.1), it is not hard to prove that the function ψ_1 defined in (3.3) has a critical point $(\mathbf{d_0}, \mathbf{t_0})$ which is a strict minimum and stable. Since,

$$\mathcal{I}_\varepsilon(\mathbf{d}, \mathbf{t}, \mathbf{v}) - c_1\beta(\xi_*)^r - \varepsilon\psi_0(\xi_*) - \varepsilon\psi_1(\mathbf{d}, \mathbf{t}) = O(\rho^2) \tag{3.6}$$

then, for every fixed $\mathbf{v} = (v_1, \ldots, v_k)$ such that

$$|v_j| \leq C \text{ and } |v_i - v_j| \geq \frac{1}{C}, i, j = 1, \cdots k, i \neq j,$$

there exists $(\mathbf{d}_\epsilon(\mathbf{v}), \mathbf{t}_\epsilon(\mathbf{v}))$ such that $\nabla_{(\mathbf{d},\mathbf{t})}\mathcal{I}_\varepsilon(\mathbf{d}_{(\epsilon,\mathbf{v})}, \mathbf{t}_{(\epsilon,\mathbf{v})}, \mathbf{v}) = 0$. Moreover, we have

$$\begin{aligned}\varepsilon\nabla_{(\mathbf{d},\mathbf{t})}\psi_1(\mathbf{d}_\epsilon(\mathbf{v}), \mathbf{t}_\epsilon(\mathbf{v})) &= \varepsilon D^2_{(\mathbf{d},\mathbf{t})}\psi_1(\mathbf{d_0}, \mathbf{t_0}) \cdot ((\mathbf{d_0}, \mathbf{t_0}) - (\mathbf{d}_\epsilon(\mathbf{v}), \mathbf{t}_\epsilon(\mathbf{v}))) + O(\varepsilon(|(\mathbf{d}_\epsilon(\mathbf{v}), \mathbf{t}_\epsilon(\mathbf{v})) - (\mathbf{d_0}, \mathbf{t_0})|^2)) \\ &= -\rho^2\nabla_{(\mathbf{d},\mathbf{t})}\psi_2(\mathbf{d}_\epsilon(\mathbf{v}), \mathbf{t}_\epsilon(\mathbf{v}), \mathbf{v}) - \nabla_{(\mathbf{d},\mathbf{t})}(\psi_3(\mathbf{d}_\epsilon(\mathbf{v}), \mathbf{t}_\epsilon(\mathbf{v}), \mathbf{v}) + o(\epsilon^2)).\end{aligned}$$

Now, using (3.5), one can get that

$$|(\mathbf{d_0}, \mathbf{t_0}) - (\mathbf{d}_\epsilon(\mathbf{v}), \mathbf{t}_\epsilon(\mathbf{v}))| = O(\frac{\rho^2}{\epsilon}). \tag{3.7}$$

On the other hand, if we consider the function

$$Q(\mathbf{v}) := \mathcal{I}_\varepsilon(\mathbf{d}_\varepsilon(\mathbf{v}), \mathbf{t}_\varepsilon(\mathbf{v}), \mathbf{v}),$$

then since $(\mathbf{d_0}, \mathbf{t_0})$ is a critical point of ψ_1, a Taylor expansion yields

$$\begin{aligned}Q(\mathbf{v}) &= c_1\beta(\xi_*)^r + \varepsilon\psi_0(\xi_*) + \varepsilon\psi_1(\mathbf{d}_\epsilon(\mathbf{v}), \mathbf{t}_\epsilon(\mathbf{v})) + \rho^2\psi_2(\mathbf{d}_\epsilon(\mathbf{v}), \mathbf{t}_\epsilon(\mathbf{v}), \mathbf{v}) + \psi_3(\mathbf{d}_\epsilon(\mathbf{v}), \mathbf{t}_\epsilon(\mathbf{v}), \mathbf{v}) + o(\epsilon^2) \\ &= c_1\beta(\xi_*)^r + \varepsilon\psi_0(\xi_*) + \varepsilon\bigg(\psi_1(\mathbf{d_0}, \mathbf{t_0}) + D^2_{\mathbf{d},\mathbf{t}}\psi_1(\mathbf{d_0}, \mathbf{t_0}) \cdot \Big((\mathbf{d_0}, \mathbf{t_0}) - (\mathbf{d}_\epsilon(\mathbf{v}), \mathbf{t}_\epsilon(\mathbf{v}))\Big)^2\bigg) \\ &+ \rho^2\psi_2(\mathbf{d}_\epsilon(\mathbf{v}), \mathbf{t}_\epsilon(\mathbf{v}), \mathbf{v}) + o(\rho^2) \\ &= c_1\beta(\xi_*)^r + \varepsilon\psi_0(\xi_*) + \varepsilon\psi_1(\mathbf{d_0}, \mathbf{t_0}) + \rho^2\psi_2(\mathbf{d}_\epsilon(\mathbf{v}), \mathbf{t}_\epsilon(\mathbf{v}), \mathbf{v}) + o(\rho^2).\end{aligned}$$

In order to prove Theorem 1.1 we will take $\lambda_i = 0$ for all $i = 1 \dots, k$, then the main term in the right hand side in the above identity becomes

$$\rho^2\bigg(-\sum_{i>j}\frac{\beta(\xi_*)^r(d^i_\epsilon(v)d^j_\epsilon(v))^{\frac{n-2}{2}}t^i_\epsilon(v)t^j_\epsilon(v)}{|v_i - v_j|^n} + \sum_j \Gamma(\xi_*, v_j)\bigg).$$

Then, assuming $\Gamma(\xi_*, v_i)$ negative definite and using (3.7), the function $Q(\mathbf{v})$ has a global maximum. This concludes the proof of Theorem 1.1.

Finally, to prove Theorem 1.3, we take $k = 2$, $\lambda_1 = 0$ and $\lambda_2 = 1$. Then, $\rho^2\psi_2(\mathbf{d}_\epsilon(\mathbf{v}), \mathbf{t}_\epsilon(\mathbf{v}), \mathbf{v})$ becomes

$$\rho^2\psi_2(\mathbf{d}_\epsilon(\mathbf{v}), \mathbf{t}_\epsilon(\mathbf{v}), \mathbf{v}) = \frac{\beta(\xi_*)^r(d^i_\epsilon(v)d^j_\epsilon(v))^{\frac{n-2}{2}}t^i_\epsilon(v)t^j_\epsilon(v)}{|v_1 - v_2|^n} + \sum_{i=1}^2 \Gamma(\xi_*, v_i).$$

Then, assuming now that $\Gamma(\xi_*, v_i)$ is positive definite and using once again (3.7), the function $Q(\mathbf{v})$ has a critical point $\mathbf{v_0}$ and Theorem 1.3 follows at once.

3.2. **Simple concentration at the boundary.** Looking for solutions to problem $(\wp_\epsilon)$ with simple concentration at the boundary is considerably less technical than multiple concentration. The procedure is very similar to the one of the proof of Theorem 1.3 in [1], we sketch the main ideas here.

Let us consider the function

$$W^\varepsilon_{d,t,\xi^0} = \beta(\xi)^{-\frac{1}{p-1}}PU_{\delta,\xi}, \tag{3.8}$$

where the dilation parameter δ will be chosen of the form

$$\delta = \varepsilon^{\frac{n-1}{n-2}}d \text{ for some } d > 0, \tag{3.9}$$

and the concentration points satisfy

$$\xi = \xi^0 + \tau\eta(\xi^0), \qquad \xi^0 \in \partial\Omega,$$

with $\tau = \varepsilon t$ for some $t > 0$.

The configuration space where the dilation parameters and the concentration points lie is given by

$$\Lambda_2 := \Big\{(d, t, \xi^0) \in (0, \infty) \times (0, \infty) \times \partial\Omega\Big\}.$$

Similarly to Lemma 2.1 we can show that if ε small enough, then for any $(d, t, \xi^0) \in \Lambda_2$ there exists $\phi^\varepsilon_{(d,t,\xi^0)}$ which chairs similar properties than the ones in (2.6) and (2.7).

As before, the reduced energy functional $\mathcal{I}_\varepsilon^2 : \Lambda_2 \longrightarrow \mathbb{R}$ is defined by

$$\mathcal{I}_\varepsilon(d,t,\xi^0) = J_\varepsilon(W^\varepsilon_{d,t,\xi^0} + \phi^\varepsilon_{d,t,\xi^0}). \tag{3.10}$$

It holds true that a parameter $(\mathbf{d},\mathbf{t},\boldsymbol{\xi^0}) \in \Lambda_2$ is a critical point of the functional $\mathcal{I}_\varepsilon^2$ if and only if the function $W^\varepsilon_{d,t,\xi^0} + \phi^\varepsilon_{d,t,\xi^0}$ is a solution to problem $(\wp_\epsilon)$.

Next we show the asymptotic expansion for the reduced energy functional $\mathcal{I}_\varepsilon^2$ in terms of the parameters (d,t,ξ^0). A straightforward computations and the result in Lemma 2.1 show that

$$J_\varepsilon(W^\varepsilon_{d,t,\xi^0} + \phi^\varepsilon_{d,t,\xi^0}) = J_\varepsilon(W^\varepsilon_{d,t,\xi^0}) + o(\epsilon)$$

for every $n \geq 3$.

In addition, taking $k = 1$ in Lemma 4.1, it is easy to see that

$$\begin{aligned}\mathcal{I}_\varepsilon^2(d,t,\xi^0) &= J_\varepsilon(W^\varepsilon_{d,t,\xi^0}) + o(\epsilon)\\ &= c_1\beta(\xi^0)^r + \epsilon\left(c_2\left(\frac{d}{t}\right)^{n-2} + c_3\nabla(\beta(\xi_*)^r)\cdot\eta(\xi_*)t_j - c_4\log(d_j)\right) + o(\epsilon).\end{aligned}$$

Now, using that there exists a non degenerate critical point $\xi_* \in \partial\Omega$ of the restriction of β to the boundary $\partial\Omega$ such that

$$\nabla(\beta(\xi_*)^{-\frac{1}{p-1}})\cdot\eta(\xi_*) > 0.$$

one can show, as in the proof of Theorem 1.3 in [1], that $\mathcal{I}_\varepsilon^2$ has a critical point. This conclude the proof.

4. Estimates on the energy

4.1. **Proof of Lemma 3.1.** The objective of this section is to give a proof of Lemma 3.1. This lemma gives a asymptotic expansion of the reduced energy functional $I_\varepsilon : \Lambda \to \mathbb{R}$ defined by

$$I(\mathbf{d},\mathbf{t},\mathbf{v}) = J_\varepsilon(V^\varepsilon_{\mathbf{d},\mathbf{t},\mathbf{v}} + \phi^\varepsilon_{\mathbf{d},\mathbf{t},\mathbf{v}})$$

in terms of the parameters $(\mathbf{d},\mathbf{t},\mathbf{v})$. Recall that for $(\mathbf{d},\mathbf{t},\mathbf{v}) \in \Lambda$ we write

$$V^\varepsilon_{(\mathbf{d},\mathbf{t},\mathbf{v})} := \sum_i (-1)^{\lambda_i} b_i P U_i,$$

where $b_i := \beta(\xi_i)^{-\frac{1}{p-1}}$ and $U_i := U_{\delta_i,\xi_i}$. Here

$$\xi_i = \xi_i^0 + \tau_i\eta(\xi_i^0) = \xi_* + \rho v_i + \eta(\xi_*)g(\rho v_i) + \tau_i\eta(\xi_i^0),$$

where $\delta_i = d_i\varepsilon^{\frac{n-1}{n-2}}$, $\tau_i = t_i\varepsilon$,

$$v_i \in T_{\xi_*}\partial\Omega := \{v \in \mathbb{R}^n : \eta(\xi_*)\cdot v = 0\},$$

$\rho = \varepsilon^{\frac{n+1}{n+2}}$ and $g : T_{\xi_*}\partial\Omega \to \mathbb{R}$ satisfies that

$$g(0) = 0 \text{ and } \nabla g(0) = 0.$$

Notice that since ξ_* is a critical point of β restricted to the boundary then

$$\nabla\beta(\xi_*)\cdot v = 0 \qquad \text{for all } v \in T_{\xi_*}\partial\Omega.$$

Note also that, for $r := -\frac{2}{p-1}$, we can write

$$\beta(\xi_i)^r = \beta(\xi_*)^r + \varepsilon\nabla(\beta(\xi_*)^r)\cdot\eta(\xi_*)t_i + \rho^2\Gamma(\xi_*,v_i) + o(\epsilon^2), \tag{4.1}$$

where $\Gamma(\xi_*,v_i)$ is defined in (3.1). Indeed, if we use a Taylor expansion

$$\begin{aligned}\beta(\xi_i)^r &= \beta(\xi_* + \rho v_i + \eta(\xi_*)g(\rho v_i) + \tau_i\eta(\xi_i^0))^r\\ &= \beta(\xi_*)^r + \nabla(\beta(\xi_*)^r)\cdot(\rho v_i + \eta(\xi_*)g(\rho v_i) + \tau_i\eta(\xi_i^0))\\ &+ D^2(\beta(\xi_*)^r)\cdot(\rho v_i + \eta(\xi_*)g(\rho v_i) + \tau_i\eta(\xi_i^0))^2 + o(\epsilon^2)\\ &= \beta(\xi_*)^r + \varepsilon\nabla(\beta(\xi_*)^r)\cdot\eta(\xi_i^0)t_i\\ &+ \rho^2\Big(\nabla(\beta(\xi_*)^r)\cdot\eta(\xi_*)(D^2g(\xi_*)v_i\cdot v_i) + D^2(\beta(\xi_*^r))v_i\cdot v_i\Big) + o(\epsilon^2)\\ &= \beta(\xi_*)^r + \varepsilon\nabla(\beta(\xi_*)^r)\cdot\eta(\xi_*)t_i + \rho^2\Gamma(\xi_*,v_i) + o(\epsilon^2),\end{aligned}$$

because

$$\eta(\xi_i^0) = \eta(\xi_*) + \mathcal{O}, \text{ where } |\mathcal{O}| = O(\rho).$$

We will denote

$$\gamma_1 = \int_{\mathbb{R}^n} U_{1,0}^{p+1}, \gamma_2 = \int_{\mathbb{R}^n} U_{1,0}^{p} \text{ and } \gamma_3 = \int_{\mathbb{R}^n} U_{1,0}^{p+1} log(U_{1,0})$$

Equation (4.1) will be use to compute the following expansions.

Lemma 4.1. *We have the following expansions*

(4.2)

$$\begin{aligned}\frac{1}{2}\int_\Omega |\nabla V^\varepsilon_{\mathbf{d},\mathbf{t},\mathbf{v}}|^2 =& \frac{\gamma_1}{2} k\beta(\xi_*)^r \\ &+\varepsilon\left(\frac{\gamma_1}{2}\nabla(\beta(\xi_*)^r)\cdot\eta(\xi_*)\sum_i^k t_i - \beta(\xi_*)^r \frac{c_n\gamma_2}{2}\sum_i^k\left(\frac{d_i}{t_i}\right)^{n-2}\right) + O(\varepsilon^{\frac{n}{n-2}})\mathcal{R}_1(d,t) \\ &+\rho^2\left(\frac{\gamma_1}{2}\sum_i^k \Gamma(\xi_*,v_i) + \gamma_2\beta(\xi_*)^r\sum_{i>j}^k(-1)^{\lambda_i}(-1)^{\lambda_j}\frac{d_i^{\frac{n-2}{2}}d_j^{\frac{n-2}{2}}t_it_j}{|v_i-v_j|^n}\right) \\ &+o(\epsilon^2)\end{aligned}$$

$$\begin{aligned}\frac{1}{p+1}\int_\Omega \beta(x)|V^\varepsilon_{\mathbf{d},\mathbf{t},\mathbf{v}}|^{p+1} &= \frac{\gamma_1}{p+1}k\beta(\xi_*)^r \\ &+ \varepsilon\left(\frac{\gamma_1}{p+1}\nabla(\beta(\xi_*)^r)\cdot\eta(\xi_*)\sum_i^k t_i - \beta(\xi_*)^r c_n\gamma_2\sum_i^k\left(\frac{d_i}{t_i}\right)^{n-2}\right) + O(\varepsilon^{\frac{n}{n-2}})\mathcal{R}_2(d,t) \\ &+ \rho^2\left(\frac{\gamma_1}{p+1}\sum_i^k\Gamma(\xi_*,v_i) + 2\gamma_2\beta(\xi_*)^r\sum_{i>j}^k(-1)^{\lambda_i}(-1)^{\lambda_j}\frac{d_i^{\frac{n-2}{2}}d_j^{\frac{n-2}{2}}t_it_j}{|v_i-v_j|^n}\right) \\ &+ o(\epsilon^2)\end{aligned} \tag{4.3}$$

and

(4.4)

$$\begin{aligned}\frac{\varepsilon}{p+1}\int_{B_j}\beta(x)|V^\varepsilon_{\mathbf{d},\mathbf{t},\mathbf{v}}|^{p+1}\log(V^\varepsilon_{\mathbf{d},\mathbf{t},\mathbf{v}}) =& -\frac{\varepsilon}{p+1}(\frac{n-2}{2})\beta(\xi_*)^r\gamma_1\sum_{i=1}^k\log(d_i) + \frac{\varepsilon}{p+1}k\beta(\xi_*)\gamma_1\log(\beta(\xi_*)^{\frac{r}{2}}) \\ &- \frac{\varepsilon}{p+1}(\frac{n-1}{2})k\beta(\xi_*)\gamma_1\log(\varepsilon) + \frac{\varepsilon}{p+1}k\beta(\xi_*)^r\gamma_3 + o(\epsilon^2),\end{aligned}$$

where $\Gamma(\xi_*,v_i)$ is defined in (3.1) and $\mathcal{R}_\ell$'s are bounded smooth functions on their arguments which does not depend on the variables $(v_1,\dots,v_k)$.

Proof. We write

$$PU_i(x) = U_i(x) - \delta_i^{\frac{n-2}{2}}H(x,\xi_i) + \Pi(x,\xi_i).$$

Lemma 5.2 shows that function $\Pi(x,\xi_i)$ satisfies

$$\begin{aligned}\Pi(\xi_i,\xi_i) &= c\frac{\delta_i^{\frac{n+2}{2}}}{\varepsilon^n}\left(\gamma_{1,1}(\frac{\text{dist}(\xi_i,\partial\Omega)}{\varepsilon}) + O(\varepsilon)\right), \\ &= c\frac{\delta_i^{\frac{n+2}{2}}}{\varepsilon^n}\left(\mathcal{R}(t_i) + O(\varepsilon)\right),\end{aligned} \tag{4.5}$$

here the function $\mathcal{R}$ is smooth on its parameters and does not depend on $(v_1,\dots,v_k)$.

We subdivide the proof into three steps.

Step 1: *Expansion of the term* $\displaystyle\int_\Omega |\nabla V^\varepsilon_{\mathbf{d},\mathbf{t},\mathbf{s}}|^2$.

We write

$$\frac{1}{2}\int_\Omega |\nabla V^\varepsilon_{\mathbf{d},\mathbf{t},\mathbf{s}}|^2 = \frac{1}{2}\sum_{i=1}^k \int_\Omega |\nabla b_i PU_i|^2 + \sum_{i>j}(-1)^{\lambda_i}(-1)^{\lambda_j} b_i b_j \int_\Omega \nabla PU_i \cdot \nabla PU_j. \tag{4.6}$$

Let

$$\begin{aligned}\int_\Omega |\nabla b_i PU_i|^2 &= b_i^2 \int_\Omega U_i^p PU_i \\ &= b_i^2 \int_\Omega U_i^{p+1} - b_i^2 \int_\Omega U_i^p \delta_i^{\frac{n-2}{2}} H(x,\xi) + \int_\Omega U_i^p \Pi(x,\xi_i) \\ &= A_{1,i} + A_{2,i} + A_{3,i}.\end{aligned}$$

Using equation (4.1), we have the estimate

$$\begin{aligned}A_{1,i} = b_i^2 \int_\Omega U_i^{p+1} &= b_i^2 \int_{\frac{\Omega-\xi_i}{\delta_i}} U_{1,0}^{p+1} \\ &= b_i^2 \gamma_1 (1 + O(\delta^n)) \\ &= \gamma_1 \beta(\xi_*)^r + \varepsilon \gamma_1 \nabla(\beta(\xi_*)^r)\cdot \eta(\xi_*) t_i + \rho^2 \Gamma(\xi_*, v_i) + o(\epsilon^2).\end{aligned} \tag{4.7}$$

On the other hand, using Lemma 5.1 and equation (4.1), we have

$$\begin{aligned}A_{2,i} = -b_i^2 \alpha_n \int_\Omega U_i^p \delta_i^{\frac{n-2}{2}} H(x,\xi_i) &= -b_i^2 \delta_i^{n-2} \int_{\frac{\Omega-\xi_i}{\delta_i}} U_{1,0}^p \alpha_n H(\delta_i y + \xi_i, \xi_i) \\ &= -b_i^2 \delta_i^{n-2} H(\xi_i,\xi_i)\gamma_2(1+O(\delta^2)) \\ &= -\gamma_2 b_i^2 \delta_i^{n-2} \frac{c_n}{|2\tau_i|^{n-2}}(1+O(\tau_i)) \\ &= -\gamma_2 \left(\frac{\delta_i}{\tau_i}\right)^{n-2} \beta(\xi_i)^r (c_n + O(\tau_i)) \\ &= -\varepsilon c_n \gamma_2 \left(\frac{d_i}{t_i}\right)^{n-2} \beta(\xi_*)^r + O(\epsilon^2).\end{aligned} \tag{4.8}$$

Moreover, using equation (4.5), we get

$$\begin{aligned}A_{3,i} = \int_\Omega U_i^p \Pi(x,\xi_i) &= \int_{\frac{\Omega-\xi_i}{\delta_i}} \delta_i^{\frac{n-2}{2}} U_{1,0}^p \Pi(\delta_i y + \xi_i, \xi_i) \\ &= \delta_i^{\frac{n-2}{2}} \Pi(\xi_i,\xi_i) \int_{\mathbb{R}^n} U_{1,0}^p (1+o(1)) \\ &= c\delta_i^{\frac{n-2}{2}} \frac{\delta_i^{\frac{n+2}{2}}}{\varepsilon^n}(\mathcal{R}(t_i) + O(\varepsilon)) \\ &= \varepsilon^{\frac{n}{n-2}} \mathcal{R}(t,d) + o(\epsilon^2)\end{aligned} \tag{4.9}$$

here the function $\mathcal{R}(t,d)$ does not depend on the variable $(v_1,\dots,v_k)$. Using equations (4.7), (4.8) and (4.9) we have

$$\begin{aligned}\frac{1}{2}\int_\Omega |\nabla b_i PU_i|^2 = \frac{\gamma_1}{2}\beta(\xi_*)^r + \varepsilon\left(\frac{\gamma_1}{2}(\nabla\beta(\xi_*)^r)\cdot\eta(\xi_*)t_i - c_n\frac{\gamma_2}{2}\left(\frac{d_i}{t_i}\right)^{n-2}\beta(\xi_*)^r\right) + O(\varepsilon^{\frac{n}{n-2}})\mathcal{R}(d,t) \\ + \frac{\rho^2}{2}\Gamma(\xi_*,v_i) + O(\epsilon^2).\end{aligned} \tag{4.10}$$

We continue with the expansion of second term in equation (4.6):

(4.11)

$$
\begin{aligned}
&b_ib_j\int_\Omega \nabla PU_i\cdot\nabla PU_j = b_ib_j\int_\Omega U_i^pPU_j\\
=&b_ib_j\int_\Omega U_i^p(U_j-\delta_n^{\frac{n-2}{2}}H(x,\xi_j)+\Pi(x,\xi_j))\\
=&b_ib_j(\delta_i\delta_j)^{\frac{n-2}{2}}\int_{\frac{\Omega-\xi_i}{\delta_i}}U_{1,0}^p\Big(\frac{a_0}{(\delta_j^2+|\delta_iy+\xi_i-\xi_j|^2)^{\frac{n-2}{2}}}-H(\delta_iy+\xi_i,\xi_j)\Big)+b_ib_j\delta_i^{\frac{n-2}{2}}\int_{\frac{\Omega-\xi_i}{\delta_i}}U_i^p\Pi(\delta_iy+\xi_i,\xi_j)\\
=&b_ib_j(\delta_i\delta_j)^{\frac{n-2}{2}}\int_{\frac{\Omega-\xi_i}{\delta_i}}U_{1,0}^p\Big(\frac{a_0}{|\xi_i-\xi_j|^{n-2}}-H(\xi_i,\xi_j)\Big)+b_ib_j\delta_i^{\frac{n-2}{2}}\int_{\frac{\Omega-\xi_i}{\delta_i}}\Pi(\delta_iy+\xi_i,\xi_j)\gamma_2\\
+&b_ib_j(\delta_i\delta_j)^{\frac{n-2}{2}}\int_{\frac{\Omega-\xi_i}{\delta_i}}U_{1,0}^p\Big(\frac{a_0}{(\delta_j^2+|\delta_iy+\xi_i-\xi_j|^2)^{\frac{n-2}{2}}}-\frac{a_0}{|\xi_i-\xi_j|^{n-2}}+\alpha_nH(\xi_i,\xi_j)-\alpha_nH(\delta_iy+\xi_i,\xi_j)\Big)\\
=&\rho^2a_0\beta(\xi_*)^r\frac{d_i^{\frac{n-2}{2}}d_j^{\frac{n-2}{2}}t_it_j}{|v_i-v_j|^n}+O(\rho^2\frac{\delta^2}{\varepsilon^2})+O(\varepsilon^{\frac{n}{n-2}})\mathcal{R}(t,d)+o(\epsilon^2),
\end{aligned}
$$

the last equality is due to the following computations

$$
\begin{aligned}
&b_ib_j(\delta_i\delta_j)^{\frac{n-2}{2}}\int_{\frac{\Omega-\xi_i}{\delta_i}}U_{1,0}^p\Big(\frac{a_0}{|\xi_i-\xi_j|^{n-2}}-H(\xi_i,\xi_j)\Big)\\
=\;& b_ib_j(\delta_i\delta_j)^{\frac{n-2}{2}}\int_{\frac{\Omega-\xi_i}{\delta_i}}U_{1,0}^pa_0\Big(\frac{1}{|\xi_i-\xi_j|^{n-2}}-\frac{1}{|\xi_i-\widetilde{\xi}_j|^{n-2}}\Big)\\
=\;& b_ib_j\delta_i^{\frac{n-2}{2}}\delta_j^{\frac{n-2}{2}}\frac{c_n}{|\xi_i^0-\xi_j^0|^{n-2}}\left((n-2)\frac{2\tau_i\tau_j}{|\xi_i^0-\xi_j^0|^2}+O\left(\max\{\tau_i,\tau_j\}\right)\right)\\
=\;& \beta(\xi_i)^{\frac{r}{2}}\beta(\xi_i)^{\frac{r}{2}}c_n\left(\rho^2\left(\frac{d_i^{\frac{n-2}{2}}d_j^{\frac{n-2}{2}}t_it_j}{|(v_i-v_j)+\rho e_*((Av_i,v_i)-(Av_j,v_j))|^n}\right)+o(\rho^2)\right)\\
=\;& \rho^2\beta(\xi_*)^rc_n\frac{d_i^{\frac{n-2}{2}}d_j^{\frac{n-2}{2}}t_it_j}{|v_i-v_j|^n}+o(\epsilon^2),
\end{aligned}
$$

and

$$
\begin{aligned}
&b_ib_ja_0(\delta_i\delta_j)^{\frac{n-2}{2}}\int_{\frac{\Omega-\xi_i}{\delta_i}}U_{1,0}^p\Big(\frac{1}{(\delta_j^2+|\delta_iy+\xi_i-\xi_j|^2)^{\frac{n-2}{2}}}-\frac{1}{|\xi_i-\xi_j|^{n-2}}\Big)\\
&=b_ib_ja_0(\delta_i\delta_j)^{\frac{n-2}{2}}\int_{\frac{\Omega-\xi_i}{\delta_i}}U_{1,0}^pO\Big(|\xi_i-\xi_j|^{-n}(|\delta_iy|^2+(\xi_i-\xi_j)\cdot(\delta_iy))\Big)\\
&=O((\delta_i\delta_j)^{\frac{n-2}{2}}\rho^{-n}(\delta^2+\rho\delta))=o(\epsilon^2)
\end{aligned}
$$

and finally

$$
\begin{aligned}
&b_ib_j(\delta_i\delta_j)^{\frac{n-2}{2}}\int_{\frac{\Omega-\xi_i}{\delta_i}}\alpha_nU_{1,0}^p(H(\xi_i,\xi_j)-H(\delta_iy+\xi_i,\xi_j))\\
&=b_ib_j(\delta_i\delta_j)^{\frac{n-2}{2}}\int_{\frac{\Omega-\xi_i}{\delta_i}}U_{1,0}^pO\Big(\frac{1}{|\xi_i-\widetilde{\xi}_j|^{n-2}}-\frac{1}{|\delta_iy+\xi_i-\widetilde{\xi}_j|^{n-2}}\Big)\\
&=b_ib_j(\delta_i\delta_j)^{\frac{n-2}{2}}\int_{\frac{\Omega-\xi_i}{\delta_i}}U_{1,0}^pO\Big(|\xi_i-\widetilde{\xi}_j|^{-n}(|\delta_iy|^2+(\xi_i-\xi_j)\cdot(\delta_iy))\Big)\\
&=o(\epsilon^2).
\end{aligned}
$$

The proof of equation (4.2) follows from equations (4.10) and (4.11).

Step 2: *Expansion of the term* $\displaystyle\int_\Omega\beta(x)|V^\varepsilon_{\mathbf{d},\mathbf{t},\mathbf{v}}|^{p+1}$.

We set $r_1 > 0$ is given by $r_1 := \frac{1}{2}\min\{\text{dist}(\xi_i, \partial\Omega) : i = 1, \dots, k\}$. Note that $r_1 = \tau_{i_0}$ for some $i_0 \in \{1, \dots, k\}$. We define the sets $B_i := B(\xi_i, r_1)$ for $i = 1, \dots, k-1$ and $B_k := \Omega \setminus \cup_{i=1}^{k-1} B_i$. Hence

$$\frac{1}{p+1}\int_\Omega \beta(x)|V^\varepsilon_{\mathbf{d},\mathbf{t},\mathbf{v}}|^{p+1} = \frac{1}{p+1}\sum_{i=1}^{k}\int_{B_i}\beta(x)|V^\varepsilon_{\mathbf{d},\mathbf{t},\mathbf{v}}|^{p+1}.$$

On each B_j we get

$$\begin{aligned}
&\frac{1}{p+1}\int_{B_j}\beta(x)|V^\varepsilon_{\mathbf{d},\mathbf{t},\mathbf{v}}|^{p+1} = \frac{1}{p+1}\int_{B_j}\beta(x)|(-1)^{\lambda_j}b_jPU_j + \sum_{i\neq j}(-1)^{\lambda_i}b_iPU_i|^{p+1}\\
= &\frac{1}{p+1}\int_{B_j}\beta(x)|b_jPU_j|^{p+1} + \int_{B_j}\beta(x)|b_jPU_j|^{p-1}((-1)^{\lambda_j}b_jPU_j)(\sum_{i\neq j}(-1)^{\lambda_i}b_iPU_i)\\
+ &O(\int_{B_j}PU_j^{p-1}PU_i^2)\\
= &C_{1,j} + C_{2,j} + o(\epsilon^2).
\end{aligned}$$

Similar computations to the ones that we made in step 1 lead us to

(4.12)

$$\begin{aligned}
C_{1,j} =&\frac{1}{p+1}\int_{B_j}\beta(x)|b_jPU_j|^{p+1} = \frac{b_j^{p+1}}{p+1}\int_{B_j}\beta(x)|U_j - \delta_j^{\frac{n-2}{2}}H(x,\xi_j) + \Pi(x,\xi_j)|^{p+1}\\
=&\frac{b_j^{p+1}}{p+1}\int_{B_j}\beta(x)U_j^{p+1} - b_j^{p+1}\int_{B_j}\beta(x)U_j^p\delta_j^{\frac{n-2}{2}}H(x,\xi_j) + b_j^{p+1}\int_{B_j}\beta(x)U_j^p\Pi(x,\xi_j)\\
&+O\bigg(\int_{B_j}U_j^{p-1}\Big(\delta_j^{\frac{n-2}{2}}H(x,\xi_j) + \Pi(x,\xi_j)\Big)^2\bigg)\\
=&\frac{b_j^{p+1}}{p+1}\int_{\frac{B_j-\xi_j}{\delta_j}}\beta(\delta_jy+\xi_j)U_{1,0}^{p+1} - b_j^{p+1}\int_{\frac{B_j-\xi_j}{\delta_j}}\beta(\delta_jy+\xi_j)U_{1,0}^p\delta_j^{n-2}H(\delta_jy+\xi_j,\xi_j)\\
&+O\big(\varepsilon^{\frac{n}{n-2}}\big)\mathcal{R}(d,t) + o(\epsilon^2)\\
=&\frac{b_j^{p+1}}{p+1}(\beta(\xi_j)\gamma_1 + \frac{\delta_i^n}{\tau^n}\mathcal{R}(t,d)) - b_j^{p+1}\beta(\xi_j)\delta_j^{n-2}H(\xi_j,\xi_j)(\gamma_2 + \frac{\delta_j^2}{\varepsilon^2}\mathcal{R}(t,d)) + O\big(\varepsilon^{\frac{n}{n-2}}\big)\mathcal{R}(d,t) + o(\epsilon^2)\\
=&\frac{b_j^2}{p+1}(\gamma_1 + \frac{\delta_i^n}{\tau^n}\mathcal{R}(t,d)) - \varepsilon c_n\gamma_2\left(\frac{d_i}{t_i}\right)^{n-2}\beta(\xi_*)^r + O\big(\varepsilon^{\frac{n}{n-2}}\big)\mathcal{R}(d,t) + o(\epsilon^2)\\
=&\frac{\beta(\xi_*)^r}{p+1}\gamma_1 + \frac{1}{p+1}\varepsilon\gamma_1\nabla(\beta(\xi_*)^r)\cdot\eta(\xi_*)t_i - \varepsilon c_n\gamma_2\left(\frac{d_i}{t_i}\right)^{n-2}\beta(\xi_*)^r\\
&+ O\big(\varepsilon^{\frac{n}{n-2}}\big)\mathcal{R}(d,t) + \frac{\rho^2}{p+1}\gamma_1\Gamma(\xi_*,v_i) + o(\epsilon^2).
\end{aligned}$$

Also

$$
\begin{aligned}
C_{2,j} =& \int_{B_j} \beta(x)|b_j PU_j|^{p-1}((-1)^{\lambda_j} b_j PU_j)(\sum_{i\neq j}(-1)^{\lambda_i} b_i PU_i) \\
=& \sum_{i\neq j}(-1)^{\lambda_i}(-1)^{\lambda_j} b_j^p b_i \int_{B_j} \beta(x)\Big(U_j - \delta_j^{\frac{n-2}{2}} H(x,\xi_j) + \Pi(x,\xi_j)\Big)^p PU_i \\
=& \sum_{i\neq j}(-1)^{\lambda_i}(-1)^{\lambda_j} b_j^p b_i \int_{B_j} \beta(x) U_j^p PU_i \\
& -p\sum_{i\neq j}(-1)^{\lambda_i}(-1)^{\lambda_j} b_j^p b_i \int_{B_j} \beta(x) U_j^{p-1} PU_i\Big(\delta_j^{\frac{n-2}{2}} H(x,\xi_j) + \Pi(x,\xi_j)\Big) \\
=& \rho^2 a_0 \beta(\xi_*)^r \sum_{i\neq j}(-1)^{\lambda_i}(-1)^{\lambda_j} \frac{d_i^{\frac{n-2}{2}} d_j^{\frac{n-2}{2}} t_i t_j}{|v_i - v_j|^n} + O(\varepsilon^{\frac{n}{n-2}})\mathcal{R}(d,t) + o(\epsilon^2)
\end{aligned} \tag{4.13}
$$

because

$$
\begin{aligned}
b_j^p b_i \int_{B_j} \beta(x) U_j^p PU_i &= b_j b_i (\delta_i\delta_j)^{\frac{n-2}{2}} G(\xi_i,\xi_j)(\gamma_2 + o(1)) + b_i b_j \delta_j^{\frac{n-2}{2}} \int_{\frac{\Omega-\xi_i}{\delta_i}} U_{1,0}^p \Pi(\delta_i y + \xi_i, \xi_j) \\
&= \rho^2 a_0 \beta(\xi_*)^r \frac{d_i^{\frac{n-2}{2}} d_j^{\frac{n-2}{2}} t_i t_j}{|v_i - v_j|^n} + O(\varepsilon^{\frac{n}{n-2}})\mathcal{R}(d,t) + o(\epsilon^2)
\end{aligned}
$$

and

$$
\begin{aligned}
& \int_{B_j} \beta(x) U_j^{p-1} PU_i\Big(\alpha_n \delta_j^{\frac{n-2}{2}} H(x,\xi_j) + O\Big(\frac{\delta_j^{\frac{n+2}{2}}}{\tau_j^n}\Big)\Big) \\
=& \ \delta_j^{n-2}(\delta_i\delta_j)^{\frac{n-2}{2}} \int_{\frac{B_j-\xi_j}{\delta_j}} \beta(\delta_j y + \xi_j) U_{1,0}^{p-1} \frac{\alpha_n H(\delta_j y + \xi_j, \xi_j) + O\Big(\frac{\delta_j^2}{\tau_j^n}\Big)}{(\delta_i^2 + |\delta_j y + \xi_j - \xi_i|^2)^{\frac{n-2}{2}}} \\
=& \ o(\epsilon^2).
\end{aligned}
$$

The proof of (4.3) follows from equations (4.12) and (4.13).

Step 3: *Expansion of the term* $\int_\Omega \beta(x)|V^\varepsilon_{\mathbf{d},\mathbf{t},\mathbf{v}}|^{p+1}\log(V^\varepsilon_{\mathbf{d},\mathbf{t},\mathbf{v}})$.
We continue with the estimation of equation (4.4), to do this we write

$$
\frac{\varepsilon}{p+1}\int_\Omega \beta(x)|V^\varepsilon_{\mathbf{d},\mathbf{t},\mathbf{v}}|^{p+1}\log(V^\varepsilon_{\mathbf{d},\mathbf{t},\mathbf{v}}) = \frac{\varepsilon}{p+1}\sum_j \int_{B_j} \beta(x)|V^\varepsilon_{\mathbf{d},\mathbf{t},\mathbf{v}}|^{p+1}\log(V^\varepsilon_{\mathbf{d},\mathbf{t},\mathbf{v}}).
$$

Working on the set B_j, we have

$$\begin{aligned}
&\frac{\varepsilon}{p+1}\int_{B_j}\beta(x)|V^{\varepsilon}_{\mathbf{d},\mathbf{t},\mathbf{v}}|^{p+1}\log(V^{\varepsilon}_{\mathbf{d},\mathbf{t},\mathbf{v}})\\
&=\frac{\varepsilon}{p+1}\int_{B_j}\beta(x)|\sum_i b_iPU_i|^{p+1}\log(\sum_{i\neq j}b_iPU_i+b_jPU_j)\\
&=\frac{\varepsilon}{p+1}\int_{B_j}\beta(x)|\sum_i b_iPU_i|^{p+1}\log\left(b_jPU_j\left(1+\frac{\sum_{i\neq j}b_iPU_i}{b_jPU_j}\right)\right)\\
&=\frac{\varepsilon}{p+1}\int_{B_j}\beta(x)|\sum_{i\neq j} b_iPU_i+b_jPU_j|^{p+1}\left(\log(b_jPU_j)+\log\left(1+\frac{\sum_{i\neq j}b_iPU_i}{b_jPU_j}\right)\right)\\
&=\frac{\varepsilon}{p+1}\int_{B_j}\beta(x)\left((p+1)\sum_{i\neq j}b_iPU_i\,(b_jPU_j)^p+(b_jPU_j)^{p+1}\right)\left(\log(b_jPU_j)+\frac{\sum_{i\neq j}b_iPU_i}{b_jPU_j}\right)\\
&+O\Big(\int_{B_j}(\sum_{i\neq j}b_iPU_i)^2\,(b_jPU_j)^{p-1}\left(\log(b_jPU_j)+\frac{\sum_{i\neq j}b_iPU_i}{b_jPU_j}\right)\Big)\\
&=Z_{1,j}+Z_{2,j}+Z_{3,j}+Z_{4,j}+o(\epsilon^2)
\end{aligned}$$

where

$$\begin{aligned}
Z_{1,j}:=&\frac{\varepsilon}{p+1}\int_{B_j}\beta(x)(b_jPU_j)^{p+1}\log(b_jPU_j)\\
Z_{2,j}:=&\frac{\varepsilon}{p+1}\int_{B_j}\beta(x)(b_jPU_j)^p\Big(\sum_{i\neq j}b_iPU_i\Big)\\
Z_{3,j}:=&\frac{\varepsilon}{p+1}\sum_{i\neq j}(p+1)\int_{B_j}\beta(x)b_iPU_i\,(b_jPU_j)^p\log(b_jPU_j)\\
Z_{4,j}:=&\frac{\varepsilon}{p+1}\sum_{i\neq j}(p+1)\int_{B_j}\beta(x)b_iPU_i\,(b_jPU_j)^p\Big(\sum_{i\neq j}b_iPU_i\Big).
\end{aligned}$$

Using arguments similar to those given in Step 1 we get

$$Z_{l,j}=O(\varepsilon^2)\qquad\text{for } l=2,3,4. \tag{4.14}$$

On the other hand

$$\begin{aligned}
Z_{1,j}&=\frac{\varepsilon}{p+1}\int_{B_j}\beta(x)(b_jPU_j)^{p+1}\log(b_jPU_j)\\
&=\frac{\varepsilon}{p+1}\int_{B_j}\beta(x)(b_jPU_j)^{p+1}\log(b_j\delta_j^{-\frac{n-2}{2}}\delta_j^{\frac{n-2}{2}}PU_j)\\
&=\frac{\varepsilon}{p+1}\log\left(b_j\delta_j^{-\frac{n-2}{2}}\right)\int_{B_j}\beta(x)|PU_j|^{p+1}+\frac{\varepsilon}{p+1}\int_{B_j}\beta(x)PU_j^{p+1}\log(\delta_j^{\frac{n-2}{2}}PU_j)\\
&=W_{1,j}+W_{2,j}.
\end{aligned}$$

Here, using (4.12), we obtain

$$\begin{aligned}
W_{1,j}=&\frac{\varepsilon}{p+1}\log\left(b_j\delta_j^{-\frac{n-2}{2}}\right)b_j^{p+1}\int_{B_j}\beta(x)|PU_j|^{p+1}\\
=&\frac{\varepsilon}{p+1}(\log(b_j)+\log(\varepsilon^{-\frac{n-1}{2}})-\frac{n-2}{2}log(d_i))\Big(\beta(\xi_*)^r\gamma_1+O(\varepsilon)\Big)\\
=&-\frac{\varepsilon}{p+1}\,\frac{n-2}{2}\,\beta(\xi_*)^r\gamma_1log(d_i)+\frac{\varepsilon}{p+1}\beta(\xi_*)\gamma_1log(\beta(\xi_*)^{\frac{r}{2}})\\
&-\frac{\varepsilon}{p+1}\,\frac{n-1}{2}\,\beta(\xi_*)\gamma_1log(\varepsilon)+O(\epsilon^2),
\end{aligned} \tag{4.15}$$

and, by (4.8) and (4.9), we get

$$\begin{aligned} W_{2,j} &= \frac{\varepsilon}{p+1} b_j^{p+1} \int_{B_j} \beta(x) P U_j^{p+1} \log(\delta_j^{\frac{n-2}{2}} P U_j) \\ &= \frac{\varepsilon}{p+1} b_j^{p+1} \int_{B_j} \beta(x) P U_j^{p+1} \log(\delta_j^{\frac{n-2}{2}} U_j) \\ &+ \frac{\varepsilon}{p+1} b_j^{p+1} \int_{B_j} \beta(x) P U_j^{p} (-\delta^{\frac{n-2}{2}} H(x,\xi_j) + \Pi(x,\xi_j)) \\ &= \frac{\varepsilon}{p+1} b_j^{2} \int_{\mathbb{R}^n} U_{1,0}^{p+1} \log(U_{1,0}) + o(\varepsilon^2) \\ &= \frac{\varepsilon}{p+1} \beta(\xi_*)^r \int_{\mathbb{R}^n} U_{1,0}^{p+1} \log(U_{1,0}) + o(\varepsilon^2). \end{aligned} \tag{4.16}$$

Therefore, by equations (4.14), (4.15) and (4.16), we obtain

$$\begin{aligned} &\frac{\varepsilon}{p+1} \int_{B_j} \beta(x) |V^\varepsilon_{\mathbf{d},\mathbf{t},\mathbf{s}}|^{p+1} \log(V^\varepsilon_{\mathbf{d},\mathbf{t},\mathbf{s}}) \\ &= -\frac{\varepsilon}{p+1} (\frac{n-2}{2}) \beta(\xi_*)^r \gamma_1 \log(d_i) + \frac{\varepsilon}{p+1} \beta(\xi_*) \gamma_1 \log(\beta(\xi_*)^{\frac{r}{2}}) \\ &- \frac{\varepsilon}{p+1} (\frac{n-1}{2}) \beta(\xi_*) \gamma_1 \log(\varepsilon) + \frac{\varepsilon}{p+1} \beta(\xi_*)^r \gamma_3 + o(\varepsilon^2), \end{aligned}$$

where $\gamma_3 := \int_{\mathbb{R}^n} U_{1,0}^{p+1} \log(U_{1,0})$. This concludes the proof of the lemma. □

Proof of Lemma 3.1. A straightforward computations show that

$$\begin{aligned} \mathcal{I}_\varepsilon(\mathbf{d},\mathbf{t},\mathbf{v}) &= J_\varepsilon(V^\varepsilon_{\mathbf{d},\mathbf{t},\mathbf{v}} + \phi^\varepsilon_{\mathbf{d},\mathbf{t},\mathbf{v}}) \\ &= \frac{1}{2} \int_\Omega |\nabla V^\varepsilon_{\mathbf{d},\mathbf{t},\mathbf{v}} + \phi^\varepsilon_{\mathbf{d},\mathbf{t},\mathbf{v}}|^2 - \frac{1}{p+1-\varepsilon} \int \beta(x) |V^\varepsilon_{\mathbf{d},\mathbf{t},\mathbf{v}} + \phi^\varepsilon_{\mathbf{d},\mathbf{t},\mathbf{v}}|^{p+1-\varepsilon} \\ &= J(V^\varepsilon_{\mathbf{d},\mathbf{t},\mathbf{v}}) + \psi_3(\mathbf{d},\mathbf{t},\mathbf{v}), \end{aligned}$$

where

$$\psi_3(\mathbf{d},\mathbf{t},\mathbf{v}) := \frac{1}{2} \int_\Omega |\phi^\varepsilon_{\mathbf{d},\mathbf{t},\mathbf{v}}|^2 + A_1(\mathbf{d},\mathbf{t},\mathbf{v}) + A_2(\mathbf{d},\mathbf{t},\mathbf{v})$$

and

$$A_1(\mathbf{d},\mathbf{t},\mathbf{v}) := -\frac{1}{p+1-\varepsilon} \int_\Omega \beta(x) (|V^\varepsilon_{\mathbf{d},\mathbf{t},\mathbf{v}} + \phi^\varepsilon_{\mathbf{d},\mathbf{t},\mathbf{v}}|^{p+1-\varepsilon} - |V^\varepsilon_{\mathbf{d},\mathbf{t},\mathbf{v}}|^{p+1-\varepsilon} - (p+1-\varepsilon) |V^\varepsilon_{\mathbf{d},\mathbf{t},\mathbf{v}}|^{p-\varepsilon} \phi^\varepsilon_{\mathbf{d},\mathbf{t},\mathbf{v}})$$

$$A_2(\mathbf{d},\mathbf{t},\mathbf{v}) := -\int_\Omega (\Delta(V^\varepsilon_{\mathbf{d},\mathbf{t},\mathbf{v}}) + \beta(x) |V^\varepsilon_{\mathbf{d},\mathbf{t},\mathbf{v}}|^{p-\varepsilon}) \phi^\varepsilon_{\mathbf{d},\mathbf{t},\mathbf{v}}.$$

Since $3 \le n \le 6$, Lemma 2.1 shows that

$$\int_\Omega |\nabla \phi^\varepsilon_{\mathbf{d},\mathbf{t},\mathbf{v}}|^2 = O(\varepsilon^2 \log(\varepsilon)^2),$$

and

$$A_1(\mathbf{d},\mathbf{t},\mathbf{v}) = O(\|\phi^\varepsilon_{\mathbf{d},\mathbf{t},\mathbf{v}}\|^2_{\frac{2n}{n-2}}) = O(\varepsilon^2 \log(\varepsilon)^2).$$

Moreover, using Hölder and Sobolev inequalities we get

$$|A_2(\mathbf{d},\mathbf{t},\mathbf{v})| \le \left\| \Delta(V^\varepsilon_{\mathbf{d},\mathbf{t},\mathbf{v}}) + \beta(x) |V^\varepsilon_{\mathbf{d},\mathbf{t},\mathbf{v}}|^{p-\varepsilon} \right\|_{\frac{2n}{n+2}} \|\phi^\varepsilon_{\mathbf{d},\mathbf{t},\mathbf{v}}\|_{\frac{2n}{n-2}} = O(\varepsilon^2 \log(\varepsilon)^2),$$

because, as in the proof of Proposition 2 in [16], we have that

$$\|\Delta(V^{\varepsilon}_{\mathbf{d},\mathbf{t},\mathbf{v}}) + \beta(x)|V^{\varepsilon}_{\mathbf{d},\mathbf{t},\mathbf{v}}|^{p-\varepsilon}\|_{\frac{2n}{n+2}} = \begin{cases} \mathcal{O}\Big(\varepsilon|\log\varepsilon| + \tau + (\frac{\delta}{\tau})^4\ \log^{\frac{2}{3}}(\frac{\delta}{\tau})\Big) & for \qquad n=6 \\ \\ \mathcal{O}\Big(\varepsilon|\log\varepsilon| + \tau + (\frac{\delta}{\tau})^{n-2}\Big) & for \quad n=3,\ 4 \ \text{ and } 5. \end{cases}$$

Therefore, we have that

$$\psi_3(\mathbf{d},\mathbf{t},\mathbf{v}) = \begin{cases} \mathcal{O}\Big(\epsilon^2\,|\log\varepsilon|^{\frac{4}{3}}\Big) & for \qquad n=6 \\ \\ \mathcal{O}\Big(\varepsilon^2|\log\varepsilon|^2\Big) & for \quad n=3,\ 4 \ \text{ and } 5. \end{cases}$$

Moreover, using equation (2.10), we can prove that

$$\big|\nabla_{(\mathbf{d},\mathbf{t})}\psi_3(\mathbf{d},\mathbf{t},\mathbf{v})\big| = O(\psi_3(\mathbf{d},\mathbf{t},\mathbf{v})).$$

The next step is to expand $J_\varepsilon(V^{\varepsilon}_{\mathbf{d},\mathbf{t},\mathbf{v}})$ in terms of the parameters $(\mathbf{d},\mathbf{t},\mathbf{v})$. An application of the Mean Value Theorem shows that

$$\begin{aligned} J_\varepsilon(V^{\varepsilon}_{\mathbf{d},\mathbf{t},\mathbf{v}}) &= \frac{1}{2}\int_\Omega |\nabla V^{\varepsilon}_{\mathbf{d},\mathbf{t},\mathbf{v}}|^2 - \frac{1}{p+1-\varepsilon}\int \beta(x)|V^{\varepsilon}_{\mathbf{d},\mathbf{t},\mathbf{v}}|^{p+1-\varepsilon} \\ &= \frac{1}{2}\int_\Omega |\nabla V^{\varepsilon}_{\mathbf{d},\mathbf{t},\mathbf{v}}|^2 - \frac{1}{p+1}\int \beta(x)|V^{\varepsilon}_{\mathbf{d},\mathbf{t},\mathbf{v}}|^{p+1} - \frac{\varepsilon}{(p+1)^2}\int_\Omega \beta(x)|V^{\varepsilon}_{\mathbf{d},\mathbf{t},\mathbf{v}}|^{p+1} \\ &+ \frac{\varepsilon}{(p+1)}\int_\Omega \beta(x)|V^{\varepsilon}_{\mathbf{d},\mathbf{t},\mathbf{v}}|^{p+1}\log(|V^{\varepsilon}_{\mathbf{d},\mathbf{t},\mathbf{v}}|) + O(\varepsilon^2). \end{aligned}$$

The proof is completed by applying Lemma 4.1.

□

5. Boundary estimates of the Green function.

In this section we establish some technical estimates we used in the previous part. Recall that we will denote by $G(x,y)$ the Green's function of the Laplace operator in Ω with zero Dirichlet boundary condition and $H(x,y)$ is its regular part, i.e.

$$G(x,y) = \frac{1}{\omega_n(n-2)|x-y|^{n-2}} - H(x,y), \tag{5.1}$$

where ω_n is the volume of the unit sphere in $\mathbb{R}^n$.

Now, for $r_0 > 0$ we denote by

$$\Omega_{r_0} := \{\xi \in \Omega :\ \mathrm{dist}(\xi,\partial\Omega) < r_0\}.$$

If we will fix a r_0 small enough then for every $\xi_i \in \Omega_{r_0}$ there exists a unique $\xi_i^0 \in \partial\Omega$ such that $\xi_i := \xi_i^0 + \tau_i\eta_i(\xi_i^0)$ where $\mathrm{dist}(\xi_i,\partial\Omega) = |\xi_i - \xi_i^0| := \tau_i$. For $\xi_i \in \Omega_{r_0}$, we will shall write

$$\widetilde{\xi_i} := \xi_i^0 - \tau_i\eta(\xi_i^0)$$

thus $\widetilde{\xi_i}$ is the reflection of ξ_i on $\partial\Omega$.

The following result can be consulted, for instance, in Lemma A_1 in [1].

Lemma 5.1. *Let $\xi_i \in \Omega_{r_0}$ and $\xi_2 \in \Omega$ then, we have that*

$$H(\xi_1^0 + \tau_1\eta(\xi_1^0), \xi_2) = \frac{c}{|\widetilde{\xi_1} - \xi_2|^{n-2}} + O\left(\frac{\tau_1}{|\widetilde{\xi_1} - \xi_2|^{n-2}}\right) \tag{5.2}$$

where c is a positive constant.

Let us fix $\xi := \xi^0 + \tau\eta(\xi^0) \in \Omega_{r_0}$ close to the boundary. Consider the function $\Pi(\cdot,\xi) : \Omega \to \mathbb{R}$ defined by

$$\Pi(x,\xi) := PU_{\delta,\xi}(x) - U_{\delta,\xi}(x) + \delta^{\frac{n-2}{2}} H(x,\xi).$$

Using the maximum principle is easy to see that

$$\sup_{x\in\Omega} |\Pi(x,\xi)| = O(\frac{\delta^{\frac{n+2}{2}}}{\tau^n}).$$

In order to prove our main results we need more information about the function $\Pi(\cdot,\xi)$. The rest of this section is devoted to give a proof of the following asymptotic expansion for the function $\Pi(\cdot,\xi)$ as $d(\xi) \to 0$.

Lemma 5.2. *For every $\alpha \in (0,1)$ we have that*

$$\Pi(\xi,\xi) := \sum_{k=1}^{\infty} a_0 c_k \frac{\delta^{\frac{n-2+4k}{2}}}{\varepsilon^{n-2+2k}} \left(\gamma_{1,k}(\frac{d(\xi)}{\varepsilon}) + \varepsilon\gamma_{2,k}(\frac{d(\xi)}{\varepsilon})\kappa(\xi^0) + O(\varepsilon^{1+\alpha}) \right)$$

for every ξ close to the boundary $\partial\Omega$. Here $\kappa(\xi^0)$ is the mean curvature of Ω at the point ξ^0, $c_k = \binom{\frac{-n+2}{2}}{k} := \frac{1}{k!}(\frac{-n+2}{2})(\frac{-n+2}{2}-1)\dots(\frac{-n+2}{2}-k+1)$, and

$$\begin{aligned}
\gamma_{1,k}(\frac{d(\xi)}{\varepsilon}) &:= \frac{2d(\xi)}{\varepsilon n\omega_n} \int_{\mathbb{R}^{n-1}} \frac{dy'}{|(y', \frac{d(\xi)}{\varepsilon})|^{2n-2+2k}}, \\
\gamma_{2,k}(\frac{d(\xi)}{\varepsilon}) &:= \frac{(n-2+2k)d(\xi)^2}{\varepsilon^2 n\omega_n} \int_{\mathbb{R}^{n-1}} \frac{y_1^2 dy'}{|(y', \frac{d(\xi)}{\varepsilon})|^{2n+2k}}.
\end{aligned}$$

In order to prove the Lemma above, we need to introduce some notation. First notice that the function $\Pi(x,\xi)$ satisfies the problem

$$\begin{cases}
-\Delta_x \Pi(x,\xi) = 0 & \text{in } x \in \Omega, \\
\Pi(x,\xi) = \displaystyle\sum_{k=1}^{\infty} a_0 c_k \frac{\delta^{\frac{n-2+4k}{2}}}{|x-\xi|^{n-2+2k}} & \text{on } x \in \partial\Omega,
\end{cases}$$

where, as before, $c_{k,i}$ is the binomial coefficient $\binom{\frac{-n+2-2k}{2}}{i}$.

In order to simplify the computations, for every $k \in \mathbb{N}$, we introduce the function $I_k(\cdot,\xi) : \Omega \to \mathbb{R}$ to be the solution of the problem:

$$\begin{cases}
-\Delta_x I_k(x,\xi) = 0 & \text{in } x \in \Omega, \\
I_k(x,\xi) = \frac{1}{|x-\xi|^{n-2+2k}} & \text{on } x \in \partial\Omega.
\end{cases}$$

The proof of Lemma 5.2 follows from

Lemma 5.3. *For every ξ close to the boundary $\partial\Omega$, we have the following expansion*

$$I_k(\xi,\xi) = \varepsilon^{-n-2k+2} \left(\gamma_{1,k}(\frac{d(\xi)}{\varepsilon}) + \varepsilon\gamma_{2,k}(\frac{d(\xi)}{\varepsilon})\kappa(\xi^0) + O(\varepsilon^{1+\alpha}) \right), \tag{5.3}$$

where $\alpha \in (0,1)$, $\kappa(\xi^0)$ is the mean curvature of Ω at the point ξ^0.

The proof of this lemma is similar to that of Lemma 4.4 in [9], we just give a sketch of the argument here for the reader's convenience: For $x \in \mathbb{R}^n$ we will write $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$. Without loss of generality we may assume that $\xi = (0, \xi_n)$, $\eta(\xi^0) = -e_n$ and

$$T_{\xi^0}\partial\Omega = \mathbb{H} := \left\{ (\xi', \xi_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \xi_n > 0 \right\}.$$

Set $c > 0$ such that

$$\partial\Omega \cap B_c(0) = \left\{ (x', x_n) : \phi(x') = x_n \right\}$$

where $\phi : T_{\xi^0}\partial\Omega \to \mathbb{R}$ satisfies $\phi(0) = 0$ and $\nabla\phi(0) = 0$. Therefore

$$\phi(x') = \frac{1}{2}(Mx', x') + O(|x'|^3) \qquad \text{as } |x'| \to 0$$

where $M := D^2\phi(0)$. As before $\Omega_\varepsilon = \Omega/\varepsilon$. We will write $\zeta := \frac{1}{\varepsilon}\xi \in \Omega_\varepsilon$ and

$$\widehat{I}_k^\varepsilon(y,\zeta) := \varepsilon^{n-2+2k} I_k(\varepsilon y, \xi).$$

This function satisfies that

$$\begin{cases} -\Delta_y \widehat{I}_k^\varepsilon(y,\zeta) = 0 & \text{in } y \in \Omega_\varepsilon, \\ \widehat{I}_k^\varepsilon(y,\zeta) = \frac{1}{|y-\zeta|^{n-2+2k}} & \text{on } y \in \partial\Omega_\varepsilon. \end{cases}$$

Next we analyze the behavior of the function $\widehat{I}_k^\varepsilon(y,\zeta)$ at $y \in \partial\Omega_\varepsilon \cap B_{\frac{c}{\varepsilon}}(0)$. A straightforward computation shows

$$\begin{aligned}
\widehat{I}_k^\varepsilon(y,\zeta) &= \frac{1}{|(y', \frac{1}{\varepsilon}\phi(\varepsilon y')) - (0,\zeta_n)|^{n-2+2k}} \\
&= \frac{1}{|(|y'|^2 + (\frac{1}{\varepsilon}\phi(\varepsilon y') - \zeta_n)^2)|^{\frac{n-2+2k}{2}}} = \frac{1}{(|y'|^2 + \zeta_n^2 - \frac{2}{\varepsilon}\phi(\varepsilon y')\zeta_n + (\frac{1}{\varepsilon}\phi(\varepsilon y'))^2)^{\frac{n-2+2k}{2}}} \\
&= \frac{1}{(|y'|^2+\zeta_n^2)^{\frac{n-2+2k}{2}}}\left(1 + \frac{(\frac{1}{\varepsilon}\phi(\varepsilon y'))^2 - \frac{2}{\varepsilon}\phi(\varepsilon y')\zeta_n}{|y'|^2+\zeta_n^2}\right)^{\frac{-n+2-2k}{2}} \\
&= \frac{1}{|(y',\zeta_n)|^{n-2+2k}}\left(\sum_{j=0}^{\infty}\binom{\frac{-n+2-2k}{2}}{j}\frac{\left((\frac{1}{\varepsilon}\phi(\varepsilon y'))^2 - \frac{2}{\varepsilon}\phi(\varepsilon y')\zeta_n\right)^j}{|(y',\zeta_n)|^{2j}}\right) \\
&= \frac{1}{|(y',\zeta_n)|^{n-2+2k}} + \varepsilon\frac{d_k\zeta_n(My',y')}{|(y',\zeta_n)|^{n+2k}} + \varepsilon^2 O\left(\frac{\zeta_n|y'|^3 + (My',y')^2}{|(y',\zeta_n)|^{n+2k}}\right)
\end{aligned}$$

here $d_k := \frac{(n-2+2k)}{2}$.

The last equation suggests us to consider the function $n_k(z,\zeta) : \mathbb{H} \to \mathbb{R}$ which satisfies the problem

$$\begin{cases} -\Delta_y n_k(z,\zeta) = 0 & \text{in } z \in \mathbb{H}, \\ n_k(z,\zeta) = s_k(z,\zeta) & \text{on } z \in \partial\mathbb{H} \end{cases} \tag{5.4}$$

where $s_k(\cdot,\zeta) : \partial H \to \mathbb{R}$ is defied by

$$s_k(z,\zeta) := \frac{1}{|(z',\zeta_n)|^{n-2+2k}} + \varepsilon\frac{d_k\zeta_n(Mz',z')}{|(z',\zeta_n)|^{n+2k}}.$$

The solution to problem (5.4) has the explicit form

$$\begin{aligned}
n_k(z,\zeta) &= \frac{2z_n}{n\omega_n}\int_{\partial\mathbb{H}}\frac{s_k(y,\zeta)}{|y-z|^n}dA(y) \\
&= \frac{2z_n}{n\omega_n}\int_{\mathbb{R}^{n-1}}\frac{dy'}{|(y',\zeta_n)|^{n-2+2k}||(y',0)-z|^n} \\
&+ \varepsilon\frac{2z_n\zeta_n d_k}{n\omega_n}\int_{\mathbb{R}^{n-1}}\frac{(My',y')dy'}{|(y',\zeta_n)|^{n+2k}|(y',0)-z|^n}.
\end{aligned} \tag{5.5}$$

We are going to use function $n_k(z,\zeta)$ to make an estimation of $\widehat{I}_k^\varepsilon(y,\zeta)$ on $\Omega_\varepsilon \cap B_{c/\varepsilon}(\zeta^0)$. To do this we need to perform a change of variables. Let $\widehat{n}_k(\cdot,\zeta) : \Omega_\varepsilon \cap B_{c/\varepsilon}(0) \to \mathbb{R}$ defined by

$$\widehat{n}_k(y,\zeta) := n_k(T(y),\zeta)$$

where $T : \Omega_\varepsilon \cap B_{c/\varepsilon}(\zeta^0) \to \mathbb{H}$ is the function

$$T(y', y_n) := (y', y_n - \frac{1}{\varepsilon}\phi(\varepsilon y')).$$

The following result estimates the difference between the functions $\widehat{I}_k(\cdot, \zeta)$ and $\widehat{n}_k(\cdot, \zeta)$ on the set $\Omega_\varepsilon \cap B_{c/\varepsilon}(\zeta^0)$.

Lemma 5.4. *For every $\alpha \in (0, 1)$ we have that*

$$\widehat{I}_k(y, \zeta) = \widehat{n}_k(y, \zeta) + O(\varepsilon^{1+\alpha})$$

uniformly on $y \in \Omega_\varepsilon \cap B_{c/\varepsilon}(\zeta^0)$.

Proof. The proof is similar to that of Lemma 4.4 in [9]. □

Finally, the proof of Lemma 5.3 follows from the previous lemma, equation (5.5) and the following computations

$$\begin{aligned}
\varepsilon^{n-2+2k} I_k(\xi, \xi) &= \widehat{I}_k(\zeta, \zeta) \\
&= \widehat{n}_k(\zeta, \zeta) + O(\varepsilon^{1+\alpha}) \\
&= \frac{2\zeta_n}{n\omega_n} \int_{\mathbb{R}^{n-1}} \frac{dy'}{|(y', \zeta_n)|^{n-2+2k} |(y', 0) - (0, \zeta_n)|^n} \\
&+ \varepsilon \frac{2d_k \zeta_n^2}{n\omega_n} \int_{\mathbb{R}^{n-1}} \frac{(My', y')dy'}{|(y', \zeta_n)|^{n+2k} |(y', 0) - (0, \zeta_n)|^n} + O(\varepsilon^{1+\alpha}) \\
&= \frac{2\zeta_n}{n\omega_n} \int_{\mathbb{R}^{n-1}} \frac{dy'}{|(y', \zeta_n)|^{2n-2+2k}} + \varepsilon \frac{2d_k \zeta_n^2}{n\omega_n} \int_{\mathbb{R}^{n-1}} \frac{(My', y')dy'}{|(y', \zeta_n)|^{2n+2k}} + O(\varepsilon^{1+\alpha}) \\
&= \frac{2\zeta_n}{n\omega_n} \int_{\mathbb{R}^{n-1}} \frac{dy'}{|(y', \zeta_n)|^{2n-2+2k}} + \varepsilon \frac{2d_k \zeta_n^2}{n\omega_n} \sum_{i=1}^{n-1} M_{ii} \int_{\mathbb{R}^{n-1}} \frac{y_1^2 dy'}{|(y', \zeta_n)|^{2n+2k}} \\
&+ O(\varepsilon^{1+\alpha}) \\
&= \gamma_{1,k}(\frac{d(\xi)}{\varepsilon}) + \varepsilon \gamma_{2,k}(\frac{d(\xi)}{\varepsilon}) \kappa(\xi^0) + O(\varepsilon^{1+\alpha})
\end{aligned}$$

because

$$\begin{aligned}
\int_{\mathbb{R}^{n-1}} \frac{(My', y')dy'}{|(y', \zeta_n)|^{2n+2k}} &= \sum_{i,j=1}^{n-1} \int_{\mathbb{R}^{n-1}} \frac{M_{ij} y_i y_j dy'}{|(y', \zeta_n)|^{2n+2k}} \\
&= \sum_{i=1}^{n-1} \int_{\mathbb{R}^{n-1}} \frac{M_{ii} y_i^2 dy'}{|(y', \zeta_n)|^{2n+2k}} \\
&= \sum_{i=1}^{n-1} M_{ii} \int_{\mathbb{R}^{n-1}} \frac{y_1^2 dy'}{|(y', \zeta_n)|^{2n+2k}}.
\end{aligned}$$

Here M_{ij} are the entries of the matrix $M := D^2\phi(0)$.

References

[1] N. Ackermann, M. Clapp, A. Pistoia, Boundary clustered layers near the higher critical exponents, J. Differential Equations, **245** (2013) 4168-4193.

[2] A. Ambrosetti, P.H. Rabinowitz, Dual variational methods in critical point theory and applications, J. Funct. Anal. **14** (1973), 349-381.

[3] A. Bahri, Y. Li, O. Rey, On a variational problem with lack of compactness: the topological effect of the critical points at infinity, Calc. Var. PDEs. **3** (1995), no. 1, 67-93.

[4] T. Bartsch, T. D'Aprile, A. Pistoia, Multi-bubble nodal solutions for slightly subcritical elliptic problems in domains with symmetries, Ann. Inst. Henri Poincaré **30** (2013) 1027-47 .

[5] T. Bartsch, A. Micheletti, A. Pistoia, On the existence and the profile of nodal solutions of elliptic equations involving critical growth, Calc. Var. Partial Differential Equations **26** (2006) 265-282.
[6] H. Brézis, L.A. Peletier, Asymptotics for elliptic equations involving critical growth, Partial differential equations and the calculus of variations, vol. I, pp. 149-192. Progr. Nonlinear Differential Equations Appl. Birkhäuser, Boston, MA 1 (1989).
[7] J. Byeon, Z. Q. Wang, On the Hénon equation: asymptotic profile of ground states. I. Ann. Inst. H. Poincaré Anal. Non Linéaire 23 (2006), no. 6, 803-828.
[8] D. Cao, S. Peng, The asymptotic behaviour of the ground state solutions for Hénon equation. J. Math. Anal. Appl. 278 (1), 1-17 (2003).
[9] J. Dávila, M. Kowalczyk, M. Montenegro, Critical points of the regular part of the harmonic Green function with Robin boundary condition, J. Funct. Anal. 255, vol. I, 1057-1101 (2008).
[10] M. del Pino, P. Felmer, M. Musso, Multi-peak solution for super-critical elliptic problems in domains with small holes, J. Differential Equations, **182** (2002) 511-540.
[11] M. del Pino · M. Kowalczyk · M. Musso, Singular limits in Liouville-type equations. Calc. Var. **2**4 (2005) 47-81.
[12] M. del Pino, M. Musso, A. Pistoia, Super critical boundary bubbling in a semilinear Neumann problem. Ann. Inst. H. Poincare Anal. Non Linéaire 22 (2005), no. 1, 45-82.
[13] Z.C. Han, Asymptotic approach to singular solutions for nonlinear elliptic equations involving critical Sobolev exponent, Ann. Inst. H. Poincaré Anal. Non Linéaire, **8** (1991) 159-174.
[14] M. Musso, A. Pistoia, Tower of bubbles for almost critical problems in general domains, J. Math. Pures Appl. **93** (2010) 1-40.
[15] S. Peng, Multiple boundary concentrating solutions to Dirichlet Problem of Hénon equation. Acta Math. Appl. Sinica 22(1), 137-162 (2006).
[16] A. Pistoia, E. Serra, Multi-peak solutions for the Hénon equation with slightly subcritical growth, Math. Z., **256** (2007), 75-97.
[17] A. Pistoia, T. Weth, Sign changing bubble tower solutions in a slightly subcritical semilinear Dirichlet problem, Ann. Inst. H. Poincaré, Anal. Non Linéaire **24** (2007) 325-340.
[18] O. Rey, Proof of two conjectures of H. Brézis, Manuscripta Math. **65** (1989) 19-37.
[19] O. Rey, Blow-up points of solutions to elliptic equations with limiting nonlinearity, Differential Integral Equations **4** (1991), no. 6, 1155-1167.
[20] J. Wei, S. Yan, Lazer-McKenna conjecture: the critical case, J. Funct. Anal. **244** (2007), no. 2, 639-667.

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