HIGHER DIMENSIONAL CATENOID, LIOUVILLE EQUATION AND ALLEN-CAHN EQUATION

OSCAR AGUDELO, MANUEL DEL PINO, AND JUNCHENG WEI

Abstract. We build a family of entire solutions to the Allen-Cahn equation in $\mathbb{R}^{N+1}$ for $N \geq 3$, whose level set approaches the higher dimensional catenoid in a compact region and has two logarithmic ends governed by the solutions to the Liouville equation.

Contents

1. Introduction

In this work we construct entire bounded solutions to the semilinear elliptic equation

$$\Delta u + u(1-u^2) = 0, \quad \text{in } \mathbb{R}^{N+1} \tag{1.1}$$

for any $N \geq 3$. Equation (1.1) is the prototype equation in the gradient theory of phase transition phenomena, developed by Allen-Cahn in [2]. Equation (1.1) also appears as the limit version of the singularly perturbed equation

$$\varepsilon^2 \Delta v + v(1-v^2) = 0, \quad \text{in } \Omega \tag{1.2}$$

via the rescaling $u(x) := v(\varepsilon x)$ in the expanding domain $\varepsilon^{-1}\Omega$, where $\varepsilon > 0$ is a small parameter and for which solutions correspond to the critical points of the energy functional

$$J_\varepsilon(v) := \frac{\varepsilon}{2}\int_\Omega |\nabla v|^2 + \frac{1}{4\varepsilon}\int_\Omega (1-v^2)^2.$$

In the typical situation in the *phase transition phenomema* modeled by equation (1.2), the function v represents the phase of a material placed in the region Ω. There are two stables phases represented by the constant functions $v = \pm 1$ each of which corresponds a single phase material and they are global minimizers of the energy J_ε. It is then more interesting to study phases in which two different materials coexist. This is equivalent to study solutions of (1.2) connecting the two stables phases ± 1. A well known result due to Modica, see [26], states that a family of local minimizers of J_ε with energy uniformly bounded must converge in L^1-sense to a function of the form

$$u_*(x) = \chi_M - \chi_{M^c}$$

where $M \subset \Omega$, χ is the characteristic function of a set and ∂M has minimal perimeter. The intuition behind this result was the keystone in the developments of the Γ-convergence during the 70ths and lead to the discovery of the deep connection between the Allen-Cahn equation and the *Theory of Minimal surfaces.* One of the most important developments regarding this connection and concerning entire solutions of equation (1.1) is the celebrated conjecture due to E.De Giorgi, see [12].

De Giorgi's Conjecture, 1978: *Level sets* $[u = \lambda]$ *of solutions* $u \in L^\infty(\mathbb{R}^{N+1})$ *to problem* (1.1) *which are monotone in one direction, must be parallel hyperplanes, at least if* $1 \leq N \leq 7$. *That is equivalent to saying that* u*depends only on one variable, i.e for some* x_0 *and some unit vector* n,

$$u(x) = w\,(t)\,, \quad t = (x - x_0) \cdot n \tag{1.3}$$

where $w(t)$ *is the solution to the one dimensional Allen-Cahn equation*

$$w'' + w(1 - w^2) = 0, \quad \text{in } \mathbb{R}, \quad w(0) = 0, \qquad w'(t) > 0, \quad w(\pm\infty) = \pm 1. \tag{1.4}$$

De Giorgi's conjecture is in analogy with *Bernstein problem* which states that minimal hypersurfaces that are graphs of entire functions of N variables must be hyperplanes. *Bernstein Problem* is known to be true up to dimension $N = 7$, see [6, 3, 11, 19]. Bombieri, DeGiorgi and Giusti in [7] proved that Bernstein's statement is false in dimensions $N \geq 8$, by constructing a minimal graph in $\mathbb{R}^8$ which is not a hyperplane.

De Giorgi's has been proven to be true for $N = 1$ by Ghoussoub and Gui in [21], for $N = 2$ by Ambrosio and Cabre in [4] and for $3 \leq N \leq 7$ by Savin in [29], under the additional assumption that

$$\lim_{x_{N+1} \to \pm\infty} u(x', x_{N+1}) = \pm 1.$$

This conjecture was recently proven to be false for $N \geq 8$ by del Pino, Kowalczyk and Wei in [16]. The authors used a large dilation of the Bombieri-De Giorgi-Giusti minimal graph constructed in dimensions $N = 8$ as model for the nodal set of the solution they constructed together with a *Lyapunov-Schmidth Reduction* procedure.

The monotonicity assumption in the De Giorgi's conjecture is related to the stability properties of the solutions of (1.1). In this regard, another type of stable solutions in high dimensions was provided by Pacard and Wei in [30], in which the nodal set of these solutions resembles a large dilation of the Simons Cone.

Stability properties of solutions $u \in L^\infty(\mathbb{R}^{N+1})$ to (1.1) are studied through their Morse Index. The Morse Index of u, $m(u)$, is defined as the maximal dimension of a vector space E of compactly supported functions ψ for which the quadratic form

$$\mathrm{Q}(\psi, \psi) := \int_{\mathbb{R}^{N+1}} |\nabla \psi|^2 - (1 - 3u^2)\psi^2 \, < \, 0, \quad \text{for } \psi \in E - \{0\}.$$

In 2009 a new family of solutions to the Allen-Cahn equation in $\mathbb{R}^3$ was found by del Pino, Kowalczyk and Wei in [13], in this case the nodal set of these solutions resembles a large dilation of a complete embedded minimal surface with finite total curvature. Example of this kind of surfaces were found by Costa, Hoffman and Meeks, see [24, 8, 23]. The assumption of finite total curvature implies that this type of surfaces have finite Morse Index, a property that is inherited by these solutions and the two Morse indexes coincide. One of these examples is the catenoidal solution having Morse index one and nodal set diverging at logarithmic rate.

In [1], Agudelo, del Pino and Wei constructed two family of unstable solutions to equation (1.1) in $\mathbb{R}^3$ both of them with nodal set having multiple connected components diverging at infinity at a logarithmic reate. The first of these families having Morse Index one with nodal set ruled by the *Toda System* in $\mathbb{R}^2$ and the other one having large Morse index and with nodal set resembling, far away from a compact set, a set of nested catenoids, having therefore a contrasting nature respect to the examples found in [13].

All the developments mentioned above show the strong connection between the Allen-Cahn equation and the Minimal surfaces theory, but this connection has been only partly explored, in particular when providing more examples of solutions to the Allen-Cahn Equation in high dimensions. On the other hand, unlike dimension $\mathbb{R}^3$ in which a large amount of examples of minimal surfaces exist and have been analyzed, in higher dimensions $\mathbb{R}^{N+1}, N \geq 3$, there are very few examples of minimal surfaces. The catenoid being among the classical ones.

In this paper we explore the connection between the higher dimensional catenoid and the equation (1.1) for $N \geq 3$. To state our result we consider M to be the $N-$dimensional catenoid, which is described by the graph of the axially symmetric functions $\pm F$ where $F = F(|y|)$ is the unique increasing axially symmetric solution to the minimal surface equation for graphs

$$\nabla \cdot \left(\frac{\nabla F}{\sqrt{1 + |\nabla F|^2}} \right) = 0, \quad |y| > 1, \quad y \in \mathbb{R}^N \tag{1.5}$$

with the initial conditions

$$F(1) = 0, \quad \partial_r F(1) = +\infty.$$

The catenoid M has asymptotically parallel flat ends, a fact that is reflected in the asymptotics of the function F

$$F(r) \,=\, T \,-\, \frac{r^{2-N}}{N-2} \,+\, \mathcal{O}_{L^\infty(\mathbb{R}^N)}\left(r^{4-3N}\right), \quad \text{as } r \to \infty \tag{1.6}$$

with

$$T := \int_1^\infty \frac{1}{\sqrt{s^{2(N-1)} - 1}}\, ds$$

and this relations in (3.2) can be differentiated, see [18] and references therein.

At a first glance, one may think that the proper choice for the approximate nodal set of the solutions predicted in our theorem would be a large dilated version of M, $M_\varepsilon = \varepsilon^{-1} M$, with $\varepsilon > 0$ small. As pointed out in [13] and in section 7 of [20], this is not an appropriate global choice. Instead, the parallel ends of M_ε must be perturbed in order to obtain a profile for the nodal set that will lead to good sizes in the error.

The rule governing this perturbation is the *Liouville Equation*

$$\varepsilon \Delta F_\varepsilon - a_0\, e^{\frac{-2\sqrt{2}\, F_\varepsilon}{\varepsilon}} = 0, \quad \text{for } |y| > R_\varepsilon \tag{1.7}$$

where for some $\bar{a} > 0$

$$R_\varepsilon \approx \bar{a}\, \varepsilon^{-1} e^{\frac{\sqrt{2} T}{\varepsilon}}, \quad \text{as } \varepsilon \to 0.$$

The constant $a_0 > 0$ is given by

$$a_0 \,=\, \|w'\|^{-2}_{L^2(\mathbb{R})} \int_{\mathbb{R}} 6(1 - w^2(t)) e^{\sqrt{2}\, t} w'(t)\, dt > 0$$

where the function $w(t)$ is the solution of (1.4), which is given explicitly by $w(t) = \tanh(t/\sqrt{2})$.

We match the functions F and F_ε in a C^1 way by considering the additional initial conditions for F_ε at $r = R_\varepsilon$

$$F_\varepsilon(R_\varepsilon) = T - \tfrac{R_\varepsilon^{2-N}}{N-2} + \mathcal{O}\left(R_\varepsilon^{4-3N}\right), \quad \partial_r F_\varepsilon(R_\varepsilon) = R_\varepsilon^{1-N} + \mathcal{O}\left(R_\varepsilon^{3(1-N)}\right)$$

from where it is possible to conclude that as $r \to \infty$

$$F_\varepsilon(r) = \frac{\varepsilon}{2\sqrt{2}} \left(\log\left(\frac{2\sqrt{2}\, a_0}{\varepsilon^2} \right) + 2\log(r) - \log\left(2(N-2)\right) \,+\, \bar{w}_\varepsilon(r) \right) \tag{1.8}$$

where the function $\bar{w}_\varepsilon(r)$ satisfies for $r > R_\varepsilon$ that

$$|r\,\partial_r \bar{w}_\varepsilon(r)| \,+\, |\bar{w}_\varepsilon(r)| \leq \begin{cases} C\left(\frac{r}{R_\varepsilon}\right)^{-\frac{N-2}{2}}, & 3 \leq N \leq 9 \\ C\,\log\left(\frac{r}{R_\varepsilon}\right)\left(\frac{r}{R_\varepsilon}\right)^{-4}, & N = 10 \\ C\left(\frac{r}{R_\varepsilon}\right)^{\lambda_+}, & N \geq 11 \end{cases} \tag{1.9}$$

where $\lambda_+ > 0$ is defined at (3.9) below.

Denote

$$x = (x', x_{N+1}) \in \mathbb{R}^N \times \mathbb{R}, \quad r(x) := |x'|,$$

and set

$$G(x') = (1 - \chi_\varepsilon(|x'|))F(x') + \chi_\varepsilon(|x'|)F_\varepsilon(\varepsilon x'), \quad |x'| > 1$$

where χ_ε is the characteristic function of the set $\mathbb{R}^N - B_{R_\varepsilon}$. Define Σ to be the revolution hypersurface, resulting from the union of the graphs of $\pm G$ and Σ_ε a ε^{-1} dilation of Σ. The hypersurface Σ_ε is a connected complete embedded orientable surface, splitting $\mathbb{R}^{N+1}$ into two connected components. We will prove the following theorem.

Theorem 1. *For every $N \geq 3$ and any sufficiently small $\varepsilon > 0$ there exist a solution u_ε to equation* (1.1) *having the asymptotics*

$$u_\varepsilon(x) = w(z)(1 + o(1)), \quad \text{as } \varepsilon \to 0$$

uniformly on every compact subset of $\mathbb{R}^N$, where z is the normal direction to the largely dilated surface Σ_ε described above and $w(t)$ is the heteroclinic solution to (1.4) *connecting ± 1.*

Even more, as for every $\varepsilon > 0$ small and $x_{N+1} \in \mathbb{R}$, as $|x'| \to \infty$

$$u_\varepsilon(x) = w\left(x_{N+1} - \varepsilon^{-1} F_\varepsilon(\varepsilon x')\right) - w\left(\varepsilon^{-1} F_\varepsilon(\varepsilon x') - x_{N+1}\right) + 1 \,+\, o(1), \qquad x = (x', x_{N+1}) \tag{1.10}$$

where $F_\varepsilon(x')$ is the radially symmetric function described in (1.8).

As we will see throughout the proof, we will be able to give a more precise description of these solutions and their nodal sets, not only as $\varepsilon \to 0$, but also at infinity.

The strategy to prove Theorem 1 consist in apply the infinite dimensional reduction method in the spirit of the works [1, 13, 16]. One of the crucial steps in this method is the choice of an accurate approximation of a solution to (1.1). In this regard, and as pointed out by Modica, it is fundamental to select properly the set where the solution is expected to change sign.

The first candidate for nodal set would be a small perturbation of the large dilation of M, $\varepsilon^{-1}M$. Since $\varepsilon^{-1}M$ has parallel ends, the approximation in the upper end will take the approximate form

$$w\left(\frac{T}{\varepsilon} + x_{N+1} + h(\varepsilon x')\right) - w\left(\frac{T}{\varepsilon} - x_{N+1} - h(\varepsilon x')\right) + 1, \quad \text{for } x = (x', x_{N+1}), \quad |x'| \to \infty, \quad x_{N+1} \in \mathbb{R}.$$

Setting $t = \frac{T}{\varepsilon} + x_{N+1} + h(\varepsilon x')$, the approximation can be written as

$$u := w(t) - w(2\varepsilon^{-1}T - 2h(\varepsilon x') + t) - 1$$

and we find that

$$u(1 - u^2) \approx 6(1 - w^2(t))e^{-\sqrt{2}t}e^{\frac{-2\sqrt{2}T}{\varepsilon}}.$$

This last expression states that the Allen-Cahn nonlinearity takes account for the interaction of parallel ends.

It follows that at main order the error created by this approximation reads as

$$-\varepsilon^2 \Delta_{\mathbb{R}^N} h\, w'(t) + 6(1 - w^2(t))e^{-\sqrt{2}t}e^{\frac{-2\sqrt{2}T}{\varepsilon}}$$

and after the reduction procedure, the reduced equation to adjust the nodal set would read as

$$\Delta_{\mathbb{R}^N} h \sim a_0 \varepsilon^{-2} e^{\frac{-2\sqrt{2}T}{\varepsilon}}$$

where the right hand side is bounded, small but has no decay. Therefore

$$h(x') \sim a_0 e^{-\sqrt{2}t} e^{\frac{-2\sqrt{2}T}{\varepsilon}} |x'|^2, \qquad |x'| \to \infty$$

which leads to a nodal set that diverges at a fast rate away from the catenoid $\varepsilon^{-1}M$, as $|x'| \to \infty$. We solve this issue by using instead of $\varepsilon^{-1}M$, the hypersurface $\varepsilon^{-1}\Sigma$.

Our second result shows a dramatic difference between the Allen-Cahn equation and the Theory of Minimal surfaces. A result proven in [28] states that for dimensions $3 \leq N \leq 9$, the catenoid M has Morse index 1. As treated in [13], the Morse Index of the catenoid is defined as the number of negative eigenvalues of the linearization of the mean curvature operator associated to M. In contrast our result states for $\varepsilon > 0$ small and with the same restriction in the dimension, these solutions are highly unstable.

Theorem 2. *Let u_ε be the solution to equation* (1.1) *constructed in Theorem 1. Then for $\varepsilon > 0$ small and for dimensions $3 \leq N \leq 9$, $m(u_\varepsilon) = +\infty$.*

As we will see from the proof of Theorem 2, in low dimensions the infinite Morse Index, comes from the asympotics described in (1.10) and the Morse Index of the function F_ε which comes into play far away. In dimensions $N \geq 10$, due to Hardy's inequality, we expect for these solutions to have Morse Index 1, the Morse Index of M. This, in view of the results in [1, 13] and since there would be no negative directions related to the function F_ε.

The paper is structured as follows: Section 2 sets up the general geometric setting need for the proof of Theorem 1. Section 3 describes the asymptotics of the hypersurface that will be the model for the nodal set of the solutions predicted in our result while section 4 deals with the solvability theory for a Jacobi-Hardy type of operator related to the reduced problem. Section 6 describes the strategy and details of the proof of the Theorem 1 which, as mentioned before, relies on an infinite dimensional reduction procedure, a method motivated by the pioneering work of [20]. Section 7 contains some technical results used in section 6 and finally the detailed proof of Theorem 2 is provided in the last section.

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2. General geometrical gackground

Throughout our discussion we adopt the following conventions. We write $\mathbb{R}^{N+1} = \mathbb{R}^N \times \mathbb{R}$ and we denote

$$x = (x', x_{N+1}) \in \mathbb{R}^{N+1}, \quad r(x', x_{N+1}) := |x'|.$$

Let Σ be a smooth orientable $N-$ dimensional embedded hypersurface in $\mathbb{R}^{N+1}$. Next, we describe the euclidean Laplacian close to the large dilation of Σ, $\varepsilon^{-1}\Sigma$, in a system of coordinates that is suitablefor the developments in subsequent sections.

For any point $y \in \Sigma$, we write

$$y = (y', y_{N+1}) \in \Sigma \subset \mathbb{R}^{N+1}$$

and we denote by $\nu : \Sigma \to S^N$ a fixed choice of the unit normal vector to the surface Σ.

It is easy to check that if $Y : \mathcal{U} \subset \mathbb{R}^N \to \mathbb{R}^{N+1}$ is a local parametrization of the surface Σ with

$$s = (s_1 \ldots, s_N) \in \mathcal{U} \to y = Y(s) \in \Sigma$$

then the map

$$X(s, z) = Y(s) \, + \, z\, \nu(s)$$

provides local coordinates in a neighborhood $\mathcal{N}$ of the form

$$\mathcal{N} := \{x = X(s,z) : |z| < \delta_0\}$$

for some fixed $\delta_0 > 0$.

System of coordinates $X(s,z)$ is known as the Fermi coordinates associated to Σ. Observe that for a point $x = X(s,z) \in \mathcal{N}$, the variable z represents the signed distance to Σ, i.e

$$|z| = \mathrm{dist}(\Sigma, x).$$

For z small and fixed such that

$$X(\mathcal{U}, z) \subset \mathcal{N}$$

we denote by Σ_z the translated surface locally parameterized by

$$X(\cdot, z) : U \to X(\mathcal{U}, z), \qquad s \in \mathcal{U} \to X(s,z) \in \mathcal{N}.$$

We denote by $g := (g_{ij})_{N\times N}$ and $g_z := (g_{z,ij})_{N\times N}$ the metrics on Σ and Σ_z respectively, with inverses $g^{-1} := (g^{ij})_{N\times N}$ and $g_z^{-1} := (g_z^{ij})_{N\times N}$. We find that

$$g_{z,ij} = g_{ij} \,+\, z\hat{a}_{ij}(s) + z^2\,\hat{b}_{ij}(s), \quad s \in \mathcal{U}, \quad |z| < \delta_0 \tag{2.1}$$

where for $i,j = s_1, \dots, s_N$

$$g_{ij} = \langle \partial_i Y; \partial_j Y\rangle$$

and

$$\hat{a}_{ij} = \langle \partial_i Y; \partial_j \nu\rangle + \langle \partial_j Y; \partial_i \nu\rangle\,, \quad \hat{b}_{ij} = \langle \partial_i \nu; \partial_j \nu\rangle\,.$$

Hence, we may write

$$g_z \,=\, g \,+\, z\,A(s) +\, z^2\,B(s)$$

where $A = (\hat{a}_{ij})_{N\times N}$, $B = (\hat{b}_{ij})_{N\times N}$.

It is well-known that in $\mathcal{N}$, we can compute the euclidean Laplacian in Fermi coordinates using the formula

$$\Delta = \partial_z^2 + \Delta_{\Sigma_z} - H_{\Sigma_z}\partial_z$$

where Δ_{Σ_z} is the Laplace-Beltrami operator of $\Sigma_z$and H_{Σ_z} its mean curvature.

The expression for Δ_{Σ_z} is given by

$$\begin{aligned}
\Delta_{\Sigma_z} &= \frac{1}{\sqrt{\det g_z}}\partial_i\left(\sqrt{\det g_{\Sigma_z}}g_z^{ij}\partial_j\right)\\
&= g_z^{ij}\partial_{ij} \,+\, \frac{1}{\sqrt{\det g_z}}\,\partial_i\left(\sqrt{\det g_z}g_z^{ij}\right)\partial_j\\
&= g_z^{ij}\,\partial_{ij} \,-\, g_z^{kl}\,\Gamma^i_{z,kl}\,\partial_i
\end{aligned} \tag{2.2}$$

and $\Gamma^i_{z,kl}$ stand for the Christoffel symbols on Σ_z and where in every term, summation is understood over repeated indexes.

Using (2.1) and (2.2), we write

$$\Delta_{\Sigma_z} = \Delta_\Sigma + \mathbb{A}_{ij}\partial_{ij} + \mathbb{B}_i\partial_i$$

where

$$\begin{aligned}
\mathbb{A}_{ij} &= g_z^{ij} - g^{ij}\\
\mathbb{B}_i &= g_z^{kl}\left[\Gamma^i_{z,kl} - \Gamma^i_{kl}\right] \,+\, \Gamma^i_{kl}\left[g_z^{kl} - g^{kl}\right].
\end{aligned}$$

From expression (2.1) it is direct to verify that for some smooth bounded functions $a_{ij(s,z)}$ and $b_i(s,z)$

$$\mathbb{A}_{ij}(s,z) \,=\, z\,a_{ij}(s,z), \quad \mathbb{B}_i(s,z) \,=\, z\,b_i(s,z), \quad \text{in } \mathcal{N}. \tag{2.3}$$

In further developments, the geometrical setting will become more specific, allowing us to provide more precise information on the asymptotic behavior of these functions.

Next, we compute the mean curvature of Σ_z, H_{Σ_z}. Denote by κ_i the principal curvatures of Σ. From the well known formula

$$H_{\Sigma_z} = \sum_{i=1}^{N} \frac{\kappa_i}{1 - z\,\kappa_i} = \sum_{i=1}^{N} \kappa_i + z\kappa_i^2 + z^2\kappa_i^3 + \cdots$$

from where we obtain

$$H_{\Sigma_z} \,=\, H_\Sigma \,+\, z|A_\Sigma|^2 \,+\, z^2\, b_{N+1}(s,z) \tag{2.4}$$

with

$$b_{N+1}(s,z) \,=\, \sum_{i=1}^{N} \kappa_i^3 \,+\, z\kappa_i^4 + \ldots$$

and H_Σ and $|A_\Sigma|$ denote respectively the mean curvature and the norm of the second fundamental form of the hypersurface Σ.

Summarizing, we have the formula for the euclidean Laplacian expressed in Fermi coordinates in $\mathcal{N}$

$$\begin{aligned}\Delta_X \;\;&=\;\; \partial_{zz} \,+\, \Delta_\Sigma \,-\, \left(H_\Sigma \,+\, z|A_\Sigma|^2\right)\partial_z \,+ \\ &\quad + z a_{ij}(s,z)\partial_{ij} \,+\, z b_i(s,z)\partial_i - z^2 b_{N+1}(s,z)\partial_z.\end{aligned} \tag{2.5}$$

Next, let us consider the dilated surface $\Sigma_\varepsilon := \varepsilon^{-1}\Sigma$ with local coordinate system given naturally by $Y_\varepsilon : \mathcal{U}_\varepsilon \to \Sigma_\varepsilon$

$$Y_\varepsilon(s) \,=\, \varepsilon^{-1} Y(\varepsilon s), \quad s \in \mathcal{U}_\varepsilon := \varepsilon^{-1}\mathcal{U}.$$

It is direct to check that the unit normal vector to Σ_ε is given by

$$\nu_\varepsilon(y) = \nu(\varepsilon y), \quad y \in \Sigma_\varepsilon$$

so that the dilated Fermi coordinates read as

$$\begin{aligned}X_\varepsilon(s,z) \;&=\; Y_\varepsilon(s) + z\nu_\varepsilon(s) \\ &=\; \varepsilon^{-1} Y(\varepsilon s) + z\,\nu(\varepsilon s)\end{aligned}$$

which provide local coordinates in the dilated neighborhood

$$\mathcal{N}_\varepsilon := \left\{ x = X_\varepsilon(s,z) \,:\, |\varepsilon z| \le \delta_0, \quad s \in \bar{\mathcal{U}}_\varepsilon \right\}$$

We also readily check that

$$\Delta_{X_\varepsilon} = \varepsilon^2 \Delta_X$$

so that, rescaling formula (2.5) we obtain

$$\begin{aligned}\Delta_{X_\varepsilon} = \partial_{zz} \,+\, \Delta_{\Sigma_\varepsilon} \,-\, \left(\varepsilon H_\Sigma \,+\, \varepsilon^2\, z\, |A_\Sigma|^2\right)\partial_z + \\ +\, \varepsilon\, z a_{ij}(\varepsilon s, \varepsilon z)\partial_{ij} + \varepsilon^2\, b_i(\varepsilon s, \varepsilon z)\partial_i + \varepsilon^3 z^2\, b_{N+1}(\varepsilon s, \varepsilon z)\,\partial_z.\end{aligned} \tag{2.6}$$

The previous computations are not enough for the proof of our results. We need to introduce a smooth bounded parameter function h defined in Σ. Abusing the notation we set in the coordinates $Y : \mathcal{U} \to \Sigma$

$$h(Y(s)) = h(s), \quad s \in \bar{\mathcal{U}}$$

and we consider the translated coordinate system

$$X_{\varepsilon,h}(s,t) = X_\varepsilon(s, t + h(\varepsilon s))$$

and the translated and dilated neighborhood

$$\mathcal{N}_{\varepsilon,h} := \left\{ x = X_{\varepsilon,h}(s,t) : |t + h(\varepsilon s)| < \frac{\delta_0}{\varepsilon} \right\}$$

Then, by setting $z = t + h(\varepsilon s)$, we can compute the euclidean Laplacian in the translated Fermi coordinates as in [1, 13], i.e

$$
\begin{aligned}
\Delta_{X_{\varepsilon,h}} \;=\; \partial_{tt} \,+\, \Delta_{\Sigma_\varepsilon} \,-\, \varepsilon H_\Sigma\, \partial_t \,-\, \varepsilon^2 |A_\Sigma|^2\, t\, \partial_t \,- \\
-\, \varepsilon^2 \left\{ \Delta_\Sigma\, h + |A_\Sigma|^2\, h \right\} \partial_t \,-\, \varepsilon\, \partial_i h \partial_{ti} \,+\, \varepsilon^2\, \partial_i h^2\, \partial_{tt} \,+\, D_{\varepsilon,h}
\end{aligned}
\tag{2.7}
$$

where

$$
\begin{aligned}
D_{\varepsilon,h} \;\;=\;\; & \varepsilon\,(t+h) a_{ij}(\varepsilon s, \varepsilon(t+h)) \left\{ \partial_{ij} \,-\, \varepsilon\, \partial_i h\, \partial_{tj} \,-\, \varepsilon^2\, \partial_{ij} h\, \partial_t \,+\, \varepsilon^2\, \partial_i h^2\, \partial_{tt} \right\} \\
& + \varepsilon^2 (t+h)\, b_i(\varepsilon s, \varepsilon(t+h)) \left\{ \partial_i \,-\, \varepsilon\, \partial_i h\, \partial_t \right\} + \varepsilon^3 (t+h)^2 b_{N+1}\, \partial_t
\end{aligned}
\tag{2.8}
$$

and where the functions H_Σ, $|A_\Sigma|^2$, h, $\partial_i h$, $\partial_{ij} h$ are evaluated at εs.

3. Approximate nodal set

In this section we describe the asymptotics of the nodal set for the family of solutions predicted in Theorem 1 and we compute some geometric quantities which are crucial for our developments.

Let $F = F(|y|)$ be the unique increasing axially symmetric solution to the minimal surface equation for graphs

$$
\nabla \cdot \left(\frac{\nabla F}{\sqrt{1 + |\nabla F|^2}} \right) = 0, \quad |y| > 1, \quad y \in \mathbb{R}^N \tag{3.1}
$$

with the initial conditions

$$
F(1) = 0, \quad \partial_r F(1) = +\infty.
$$

From the symmetry of F, equation (3.1) can be written as the ODE

$$
r^{N-1} \partial_r F - (1 + |\partial_r F|^2)^{\frac{1}{2}} = 0, \quad r > 1
$$

from where we directly obtain that $F(r)$ is strictly increasing and $F(r) > 0$ for every $r > 1$. We also obtain the asymptotics

$$
F(r) = \begin{cases} \sqrt{\frac{2(r-1)}{N-1}} \left(1 + \mathcal{O}_{L^\infty(\mathbb{R}^N)}(r-1) \right), & \text{as } r \to 1 \\[2ex] T \,-\, \frac{r^{2-N}}{N-2} \,+\, \mathcal{O}_{L^\infty(\mathbb{R}^N)} \left(r^{4-3N} \right), & \text{as } r \to \infty \end{cases} \tag{3.2}
$$

with

$$
T := \int_1^\infty \frac{1}{\sqrt{s^{2(N-1)} - 1}}\, ds
$$

and relations in (3.2) can be differentiated for $r > 1$.

Consider the smooth hypersurfaces $M_\pm$, parameterized by

$$
y \in D \mapsto (\, y,\, \pm F(y)\,) \in \mathbb{R}^N \times \mathbb{R}
$$

respectively and where $D := \mathbb{R}^N - B_1$.

The hypersurface $M = M_+ \cup M_-$ is the $N-$dimensional catenoid and from (3.1) it is a minimal surface of revolution. An important consequence from (3.2) is that M has two almost flat parallel ends.

As mentioned above and as pointed out in [13] and in section 7 of [20], a large dilated version of M, $M_\varepsilon = \varepsilon^{-1} M$, with $\varepsilon > 0$ small must be perturbed in order to obtain the right profile for the nodal set that will lead to good sizes in the error.

To do so, we consider $\varepsilon > 0$ small and fixed and we also fix a large radius R_ε *to be chosen appropriately.* Next step consists in fixing a particular *smooth radial solution* F_ε to the Liouville equation

$$
\varepsilon \Delta F_\varepsilon - a_0\, e^{\frac{-2\sqrt{2}\, F_\varepsilon}{\varepsilon}} = 0, \quad \text{for } |y| > R_\varepsilon \tag{3.3}
$$

where

$$a_0 := \|w'\|^{-2}_{L^2(\mathbb{R})} \int_{\mathbb{R}} 6(1-w^2(t))e^{\sqrt{2}\,t}w'(t)\,dt > 0.$$

It is enough to match in a C^1 topology the catenoid M with the graphs of F_ε and $-F_\varepsilon$ outside of a large compact subset of M, determined by the large radius R_ε. Taking into account the asymptotics in (3.2) for M, we consider (3.3) with initial conditions for F_ε at $r = R_\varepsilon$, namely

$$\begin{aligned} F_\varepsilon(R_\varepsilon) &= T - \tfrac{R_\varepsilon^{2-N}}{N-2} + \mathcal{O}\left(R_\varepsilon^{4-3N}\right) \\ \partial_r F_\varepsilon(R_\varepsilon) &= R_\varepsilon^{1-N} + \mathcal{O}\left(R_\varepsilon^{3(1-N)}\right). \end{aligned}$$

The size of the radius R_ε and the asymptotic behavior of the function F_ε, both in terms of ε, can be found by using the scaling

$$F_\varepsilon(y) = \frac{\varepsilon}{2\sqrt{2}}\left[\log\left(\frac{2\sqrt{2}\,a_0\,R_\varepsilon^2}{\varepsilon^2}\right) + f\left(\frac{|y|}{R_\varepsilon}\right)\right]$$

so that

$$\Delta f - e^{-f} = 0, \quad |y| > 1, \quad y \in \mathbb{R}^N.$$

The initial conditions for F_ε translate into the initial conditions for f

$$\begin{aligned} f(1) &= \frac{2\sqrt{2}\,T}{\varepsilon} - \log\left(\frac{2\sqrt{2}\,a_0\,R_\varepsilon^2}{\varepsilon^2}\right) - \frac{2\sqrt{2}\,a_0\,R_\varepsilon^{2-N}}{\varepsilon(N-2)} + \mathcal{O}\left(R_\varepsilon^{4-3N}\varepsilon^{-1}\right) \\ \partial_r f(1) &= 2\sqrt{2}\,\varepsilon^{-1}\left(R_\varepsilon^{2-N} + \mathcal{O}\left(R_\varepsilon^{4-3N}\right)\right). \end{aligned}$$

Using the dependence on $\varepsilon > 0$ of the initial conditions, the Intermediate Value Theorem and the Implicit Function Theorem, for every $\varepsilon > 0$ small there exists a radius R_ε with asymptotics described as

$$R_\varepsilon = \bar{a}\,\varepsilon\,e^{\frac{\sqrt{2}T}{\varepsilon}}\left(1 + \mathcal{O}\left(\varepsilon^{1-N}e^{-\frac{\sqrt{2}(N-2)T}{\varepsilon}}\right)\right), \quad \text{as } \varepsilon \to 0 \tag{3.4}$$

for some positive constant $\bar{a} > 0$ and such that $f(1) = 0$. Consequently we find that

$$\partial_r f(1) = \varepsilon^{-1}\left(R_\varepsilon^{2-N} + \mathcal{O}\left(R_\varepsilon^{4-3N}\right)\right)$$

and we directly check that

$$F_\varepsilon(y) = T - \frac{R_\varepsilon^{2-N}}{N-2} + \mathcal{O}\left(R_\varepsilon^{4-3N}\right) + \frac{\varepsilon}{2\sqrt{2}}\,f\left(\frac{|y|}{R_\varepsilon}\right).$$

Thus, it remains to analyze the asymptotic behavior of solutions to the problem

$$\Delta f - e^{-f} = 0, \quad \text{for } |y| > 1, \quad f(1) = \bar{f}, \quad \partial_r f(1) = \delta \tag{3.5}$$

in the class of radially symmetric functions and where $\delta > 0$ is small.

We pass to the Emden-Fowler variable $\mathrm{s} = \log(r)$ and we set $f(r) = v(\mathrm{s})$ to find that

$$v''(\mathrm{s}) + (N-2)v'(\mathrm{s}) - e^{2\mathrm{s}-v(\mathrm{s})} = 0, \quad \mathrm{s} > 0, \quad v(0) = \bar{f}, \quad v'(0) = \delta. \tag{3.6}$$

Classical ODE theory inmplies that the initial value problem has always a solution in a maximal interval $(0, S_*)$. Let us now assume that

$$v(\mathrm{s}) = 2\mathrm{s} + \log\left(\frac{1}{2(N-2)}\right) + w(\mathrm{s}), \quad 0 < \mathrm{s} < S_*$$

so that

$$w''(\mathrm{s}) + (N-2)w'(\mathrm{s}) - 2(N-2)\left[e^{-w(\mathrm{s})} - 1\right] = 0, \quad 0 < \mathrm{s} < S_* \tag{3.7}$$

$$w(0)=\bar{f}-\log\left(\frac{1}{2(N-2)}\right),\quad w'(0)=-(2-\delta).$$

Considering the *Lyapunov functional*

$$L(\mathrm{s})=\frac{1}{2}|w'(\mathrm{s})|^2+2(N-2)e^{-w(\mathrm{s})}+2(N-2)w(\mathrm{s}),\quad \mathrm{s}\in(0,S_*)$$

we find that

$$\frac{d}{d\mathrm{s}}L(\mathrm{s})=-(N-2)|w'(\mathrm{s})|^2\leq 0$$

and since $N\geq 3$, it follows that

$$L(\mathrm{s})\leq L(0)\leq C$$

where $C>0$ does not depend on $\delta>0$ small. This implies that $w(\mathrm{s})$ and $w'(\mathrm{s})$ remain uniformly bounded and so they are defined for every $\mathrm{s}>0$.

About the asymptotics of $w(\mathrm{s})$, we proceed as in section 2 in [10]. Observe that we can write equation (3.7) as

$$w''+(N-2)w'+2(N-2)w-N(w)=0,\quad \mathrm{s}>0 \tag{3.8}$$

$$N(w)=2(N-2)\left[e^{-w}-1+w\right].$$

The indicial roots associated to the linear part in (3.8) are the solutions to the algebraic equation

$$\lambda^2+(N-2)\lambda+2(N-2)=0$$

from where

$$\lambda_\pm\ =-\frac{N-2}{2}\pm\frac{1}{2}\sqrt{(N-2)(N-10)}. \tag{3.9}$$

Hence, a fundamental setfor the linear part of equation (3.8) is given by

$$w_+(\mathrm{s}):=e^{-\frac{N-2}{2}\mathrm{s}}\cdot\begin{cases}\cos\left(\frac{1}{2}\sqrt{(N-2)(10-N)}\,\mathrm{s}\right), & 3\leq N\leq 9\\[2ex] 1, & N=10\\[2ex] e^{+\frac{1}{2}\sqrt{(N-2)(N-10)}\,\mathrm{s}}, & N\geq 11\end{cases} \tag{3.10}$$

$$w_-(\mathrm{s})=e^{-\frac{N-2}{2}\mathrm{s}}\cdot\begin{cases}\sin\left(\frac{1}{2}\sqrt{(N-2)(10-N)}\,\mathrm{s}\right), & 3\leq N\leq 9\\[2ex] \mathrm{s}, & N=10\\[2ex] e^{-\frac{1}{2}\sqrt{(N-2)(N-10)}\,\mathrm{s}}, & N\geq 11\end{cases} \tag{3.11}$$

with Wronski determinant

$$\mathrm{W}(\mathrm{s}):=\begin{cases}\frac{1}{2}\sqrt{(N-2)(10-N)}\,e^{-\frac{N-2}{2}\mathrm{s}}, & 3\leq N\leq 9\\[2ex] e^{-8\,\mathrm{s}}, & N=10\\[2ex] \frac{1}{2}\sqrt{(N-2)(N-10)}\,e^{-\frac{N-2}{2}\mathrm{s}}, & N\geq 11.\end{cases} \tag{3.12}$$

Therefore, we can write

$$w(\mathrm{s})=A\,w_+(\mathrm{s})+B\,w_-(\mathrm{s})+\bar{w}(\mathrm{s})$$

where $\bar{w}$ is a particular solution to the equation

$$\bar{w}''+(N-2)\bar{w}'+2(N-2)\bar{w}=-N(w),\quad \mathrm{s}>0.$$

We only describe the resonant case $N = 10$, since the other cases can be described in a similar fashion. Using variations of parameters formula when $N = 10$, we choose $\bar{w}$ to be the function

$$\bar{w}(s) = e^{-4\,s}\int_s^\infty \tau e^{4\,\tau} N(w)d\tau - s\,e^{-4\,s}\int_s^\infty e^{4\,\tau}N(w)d\tau.$$

Using a fixed point argument, the following Lemma readily follows.

Lemma 3.1. *The function v solving equation* (3.6) *has the following asymptotic behavior*

$$v(\mathrm{s}) = 2\mathrm{s} + \log\left(\frac{1}{2(N-2)}\right) + w(\mathrm{s}), \quad 0 < \mathrm{s} < \infty$$

where the function $w(\mathrm{s})$ satisfies, as $\mathrm{s} \to \infty$

$$|w(\mathrm{s})| \le \begin{cases} C\,e^{-\frac{N-2}{2}\mathrm{s}}, & 3 \le N \le 9 \\ C\,\mathrm{s}\,e^{-4s}, & N = 10 \\ C\,e^{\lambda_+\,\mathrm{s}}, & N \ge 11 \end{cases} \tag{3.13}$$

where

$$\lambda_+ \;=\; -\frac{N-2}{2} \;+\; \frac{1}{2}\sqrt{(N-2)(N-10)} \;<\; 0.$$

and these relations can be differentiated in $s > 0$.

Lemma 3.1 can be restated for the function $f(r) = v(\log(r))$ as follows.

Corollary 3.1. *The function f solving problem* (3.5) *has the asymptotics as $r \to \infty$*

$$f(r) = 2\log(r) - \log\left(2(N-2)\right) + \bar{\omega}(r)$$

where

$$|r\,\partial_r\,\bar{\omega}(r)| \;+\; |\bar{\omega}(r)| \le \begin{cases} C\,r^{-\frac{N-2}{2}}, & 3 \le N \le 9 \\ C\,\log(r)\,r^{-4}, & N = 10 \\ C\,r^{\lambda_+}, & N \ge 11 \end{cases} \tag{3.14}$$

Summarizing, for every $\varepsilon > 0$ small

$$F_\varepsilon(r) = T - \frac{R_\varepsilon^{2-N}}{N-2} + \mathcal{O}\left(R_\varepsilon^{4-3N}\right) + \frac{\varepsilon}{2\sqrt{2}} f\left(\frac{|y|}{R_\varepsilon}\right), \quad r > R_\varepsilon$$

and as $r \to \infty$

$$F_\varepsilon(r) = \frac{\varepsilon}{2\sqrt{2}}\left(\log\left(\frac{2\sqrt{2}\,a_0}{\varepsilon^2}\right) + 2\log(r) - \log\left(2(N-2)\right) \,+\, \bar{\omega}_\varepsilon(r)\right). \tag{3.15}$$

where the function $\bar{\omega}_\varepsilon(r)$ satisfies for $r > R_\varepsilon$ that

$$|r\,\partial_r\bar{\omega}_\varepsilon(r)| \;+\; |\bar{\omega}_\varepsilon(r)| \le \begin{cases} C\left(\frac{r}{R_\varepsilon}\right)^{-\frac{N-2}{2}}, & 3 \le N \le 9 \\ C\,\log\left(\frac{r}{R_\varepsilon}\right)\left(\frac{r}{R_\varepsilon}\right)^{-4}, & N = 10 \\ C\left(\frac{r}{R_\varepsilon}\right)^{\lambda_+}, & N \ge 11 \end{cases} \tag{3.16}$$

these relations can be differentiated in $r > R_\varepsilon$ and where we recall from (3.4) that

$$R_\varepsilon \approx \bar{a}\,\varepsilon^{-1}e^{\frac{\sqrt{2}T}{\varepsilon}}, \quad \text{as } \varepsilon \to 0.$$

Consider now a characteristic function $\chi : \mathbb{R}_+ \to [0,1]$ such that

$$\chi(s) = \begin{cases} 0, & s < 1 \\ 1, & s > 1 \end{cases}$$

and denote $\chi_\varepsilon(r) = \chi(\frac{r}{R_\varepsilon})$.

As predicted abov, we glue the catenoid M with the graphs of $\pm F_\varepsilon$ by means of the axially symmetric function $G(r)$ described by

$$G(r) \,=\, (1-\chi_\varepsilon)\,F(r) \,+\, \chi_\varepsilon\, F_\varepsilon(r), \quad r>1. \tag{3.17}$$

Let Σ_+ be the graph of the function $+G$ described by

$$\Sigma_+ = \{(x', x_{N+1}): \quad x_{N+1} = +G(x'), \quad |x'|>1\}\,.$$

Analogously we define Σ_- as the graph of $-G$. Define also

$$\Sigma := \Sigma_+ \cup \Sigma_-.$$

For notational simplicity we have omitted the explicit dependence on $\varepsilon>0$ of G and Σ. Notice also that $\bar{\Sigma}$ is a C^1 connected hypersurface of revolution having the $x_{N+1}-$axis as its axis of symmetry and dividing $\mathbb{R}^{N+1}$ into two connected components say S_+ and S_-, where we choose S_+ to be the component containing the axis of symmetry.

We consider the local parametrization for $\Sigma_\pm$ in polar coordinates

$$Y_\pm(r,\Theta) = (\, r\Theta\,,\, \pm G(r)\,), \quad Y_\pm : (1,+\infty)\times S^{N-1} \to \mathbb{R}^{N+1}.$$

Abusing the notation, we know that the normal vector to $\Sigma_\pm$, $\nu_\pm : \Sigma_\pm \to S^N$ pointing towards the $x_{N+1}-$axis and in terms of the local coordinates $(\,r,\Theta\,)$ is given by

$$\nu_\pm(r,\Theta) \,=\, \frac{\pm 1}{\sqrt{1+|\partial_r G(r)|^2}}\,(-\,\partial_r G(r)\,\Theta\,,\,1), \quad r>1$$

respectively in $\Sigma_\pm$ and we have associated Fermi coordinates

$$X_\pm(r,\Theta,z) \,=\, Y_\pm(r,\Theta) \,+\, z\,\nu_\pm(r,\Theta)$$

for $r>1$ and $\Theta \in S^N$.

It is not hard to check using (3.2) and (3.15) that $X_\pm$ provide local diffeomorphism in a neighborhood of the form

$$\mathcal{N}_\pm := \left\{ X_\pm(r,\Theta,z) : |z| \le \delta_0\ (1-\eta_\varepsilon) + \eta_\varepsilon F_\varepsilon(r), \quad r>1, \quad \Theta \in S^{N-1} \right\}$$

where the constant $\delta_0 = \delta_0(M) > 0$ depends only on the catenoid M.

Also we remark that the function G is not only C^1, but rather C^2 except at the point $r = R_\varepsilon$. From this, it is not difficult to verify that formula (2.5) also holds almost everywhere in this neighborhood.

Since, we are working in an axially symmetric setting we know that all the geometric quantities mentioned in section 2 share also this symmetry. In particular we know that

$$\begin{aligned}\Delta_{\Sigma_\pm} \;&=\; \frac{1}{r^{N-1}\sqrt{1+|\partial_r G(r)|^2}}\,\partial_r\left(\frac{r^{N-1}}{\sqrt{1+|\partial_r G(r)|^2}}\partial_r\right)\\ &=\; \frac{\partial_{rr}}{1+|\partial_r G(r)|^2} + \left(\frac{N-1}{r} - \frac{\partial_r G(r)}{1+|\partial_r G(r)|^2}\right)\partial_r.\end{aligned} \tag{3.18}$$

Also the principal curvatures of Σ are given by

$$\kappa_1 \,=\, \cdots \,=\, \kappa_{N-1} \,=\, \frac{\partial_r G(r)}{r\sqrt{1+|\partial_r G(r)|^2}}, \quad \kappa_N \,=\, \frac{\partial_{rr} G(r)}{(1+|\partial_r G(r)|^2)^{\frac{3}{2}}} \tag{3.19}$$

and

$$|A_\Sigma|^2 \,=\, \sum_{j=1}^{N} \kappa_j^2 \,=\, \frac{(N-1)\,[\partial_r G(r)]^2}{r^2\ (1+|\partial_r G(r)|^2)} \,+\, \frac{[\partial_{rr} G(r)]^2}{(1+|\partial_r G(r)|^2)^3}. \tag{3.20}$$

Hence, in the coordinates $Y_{\pm}(r,\Theta)$ the Jacobi operator for the surface Σ takes the form

$$\begin{aligned}\mathcal{J}_{\Sigma} &= \Delta_{\Sigma_{\pm}} + |A_{\Sigma_{\pm}}|^2 \\ &= \frac{\partial_{rr}}{1+|\partial_r G(r)|^2} + \left(\frac{N-1}{r} - \frac{\pm\,\partial_r G(r)}{1+|\partial_r G(r)|^2}\right)\partial_r + \\ &\qquad + \frac{(N-1)[\partial_r G(r)]^2}{r^2\,(1+|\partial_r G(r)|^2)} + \frac{[\partial_{rr}\,G(r)]^2}{(1+|\partial_r G(r)|^2)^3}\end{aligned} \tag{3.21}$$

and we recall that we are omitting the explicit dependence of all the quantities in $\varepsilon>0$.

Next, we consider a dilated version of the geometric objects described above. For $\varepsilon>0$ small consider the hypersurfaces $\Sigma_{\varepsilon,\pm} := \varepsilon^{-1}\Sigma_{\pm}$ which can be naturally parameterized by

$$Y_{\varepsilon,\pm}(r,\Theta,) = (\,r\,\Theta\,,\,\pm\varepsilon^{-1}G(\varepsilon r)\,),\quad r>\varepsilon^{-1},\quad \Theta\in S^{N-1}.$$

from where we get the associated Fermi coordinates

$$\begin{aligned}X_{\varepsilon,\pm}(r,\Theta) &= Y_{\varepsilon,\pm}(r,\Theta) \,+\, z\,\nu_{\varepsilon,\pm}(r,\Theta) \\ &= \frac{1}{\varepsilon}\,Y_{\pm}(\varepsilon r,\Theta) \,+\, z\,\nu_{\pm}(\varepsilon r,\Theta),\quad r>\varepsilon^{-1},\quad \Theta\in S^{N-1}.\end{aligned}$$

Observe that in this case the dilated Fermi coordinates $X_{\varepsilon,\pm}(r,\Theta,z)$, provide a local diffeomorphism in a region of the form

$$\mathcal{N}_{\varepsilon,\pm} = \left\{x = X_{\varepsilon,\pm}(r,\Theta,z) : |z| < \frac{\delta_0}{\varepsilon}\,(1-\eta_{\varepsilon}(\varepsilon r)) \,+\, \eta_{\varepsilon}(\varepsilon r)\varepsilon^{-1}F_{\varepsilon}(\varepsilon r)\right\}. \tag{3.22}$$

Next we consider a smooth bounded smooth axially symmetric function $h:\Sigma\to\mathbb{R}$ satisfying the apriori estimate

$$\|h\|_* := \|D^2 h\|_{p,2+\beta} \,+\, \|(1+r(y))^{1+\beta}D_{\Sigma}h\|_{L^\infty(\Sigma)} + \|(1+r(y))^{\beta}h\|_{L^\infty(\Sigma)} \le C\varepsilon^{\tau} \tag{3.23}$$

for some $p>1$, $\tau>0$ independent of $\varepsilon>0$ and where

$$\|D^2 h\|_{p,2+\beta} := \sup_{y\in\Sigma}(1+r(y)^{2+\beta})\|D^2 h\|_{L^p(S_{\Sigma}(y;1))}$$

where $S_{\Sigma}(y;1)$ is the cylinder of width 2 and centered around the sphere passing through $y\in\Sigma$.

We also impose the assumption that h is even over Σ, which is equivalent to saying that, in the coordinates $Y_{\pm}(r,\Theta)$,

$$h(r) := h(Y_+(r,\Theta)) = h(Y_-(r,\Theta)),\quad r>1,\quad \theta\in S^{N-1}.$$

Using the translated Fermi coordinates

$$X_{\varepsilon,h}(r,\Theta,t) := X_{\varepsilon}(r,\Theta,t+h(\varepsilon r))$$

and carrying out computations similar to those in section 2 of [1], we find that the Euclidean Laplacian in formula (2.8), when applied to axially symmetric functions, takes the form

$$\begin{aligned}\Delta_{X_{\varepsilon,h}} &= \partial_{tt} + \Delta_{\Sigma_{\varepsilon}} \\ &\quad -\varepsilon H_{\Sigma}\,\partial_t - \varepsilon^2|A_{\Sigma}|^2 t\partial t - \varepsilon^2\left\{\Delta_{\Sigma}h + |A_{\Sigma}|^2 h\right\}\partial_t \\ &\quad -2\varepsilon\partial_r h(\varepsilon r)\partial_{tr} - \varepsilon^2[\partial_r h(\varepsilon r)]^2\partial_{tt} + D_{\varepsilon,h}\end{aligned} \tag{3.24}$$

where $D_{\varepsilon,h}$ is given by

$$\begin{aligned}D_{\varepsilon,h} &= \varepsilon(t+h)a_1(\varepsilon r,\varepsilon(t+h))\,(\partial_{rr} - 2\varepsilon\partial_r h(\varepsilon r)\partial_{rt} - \varepsilon^2\partial_{rr}h(\varepsilon r)\partial_t + \alpha^2[\partial_r h(\varepsilon r)]^2\partial_{tt}) \\ &+ \varepsilon^3(t+h)a_2(\varepsilon r,\varepsilon(t+h))\partial_{ij} \\ &+ \varepsilon^2(t+h)b_1(\varepsilon r,\alpha(t+h))\,(\partial_r - \varepsilon\partial_r h(\varepsilon r)\partial_t) \\ &+ \varepsilon^3(t+h)^2 b_2(\varepsilon r,\varepsilon(t+h))\,\partial_t\end{aligned} \tag{3.25}$$

and from (2.3) (2.4) and (2.5), the smooth functions a_1,a_{ij}, b_i and have the asymptotics

$$a_1(r,z) = \mathcal{O}(r^{-2}), \quad a_{ij}(r,z) = \mathcal{O}(r^{-4}), \quad i,j = s_1,\dots,s_{N-1}$$
$$b_1(r,z) = \mathcal{O}(r^{-3}), \quad b_2(r,z) = \mathcal{O}(r^{-6})$$

as $r \to \infty$.

We finish this section with the following remark. For $x = (x', x_{N+1}) \in \mathcal{N}_{\varepsilon,\pm}$ we may write

$$x_{N+1} = X_{\varepsilon,\pm}(r,\Theta,z)\cdot e_{N+1}$$

so that

$$x_{N+1} = \pm\,\varepsilon^{-1}G(\varepsilon r) \,+\, \frac{\pm\, z}{\sqrt{1+|\partial_r G(\varepsilon r)|^2}}, \quad x \in \mathcal{N}_{\varepsilon,\pm}$$

which can be rewritten as

$$z = \pm\sqrt{1+|\partial_r G(\varepsilon r)|^2}\left(x_{N+1} \mp \varepsilon^{-1}G(\varepsilon r)\right) \tag{3.26}$$

so that, from (3.15), as $\varepsilon r \to \infty$ we have that

$$z \approx \pm\left(x_{N+1} \mp \log\left(\frac{2\sqrt{2}\,a_0}{\varepsilon^2}\right) \mp 2\log(\varepsilon r) \pm \log\left(2(N-2)\right)\right).$$

4. Jacobi Operator

An important part our developments is to solve the linear equation

$$\mathcal{L}_\varepsilon(h) \,:=\, \varepsilon^2\left(\Delta_\Sigma h + |A_\Sigma|^2 h\right) \,+\, 2\sqrt{2}a_0\chi_\varepsilon e^{-2\sqrt{2}\varepsilon^{-1}F_\varepsilon}\, h \,=\, \varepsilon^2 g, \quad \text{in } \Sigma \tag{4.1}$$

in the class of axially symmetric functions and where the function g satisfies that

$$\|g\|_{p,\beta} := \sup_{y\in\Sigma}(1+r(y))^\beta\,\|g\|_{L^p(S_\Sigma(y;1))} < \infty$$

for $\beta > 0$, $p > 1$ and where $S_\Sigma(y;1)$ is the annulus in Σ centered at $y \in \Sigma$ and width two.

Since F_ε is asymptotically logarithmic in $\varepsilon > 0$ and $r > 1$, and the $N \geq 3$, Σ might be consider as a mild perturbation of the $N-$catenoid M and therefore the size of $\mathcal{L}_\varepsilon^{-1}$ is expected to be uniformly bounded in $\varepsilon > 0$. This is precisely the content of the following Proposition.

Proposition 4.1. *For any p and β with $p > 1$ and $0 < \beta < N-2$ there exists a constant $C = C(p,\beta) > 0$ such that for every $\varepsilon > 0$ and for any axially symmetric function g defined in Σ, that is even respect to the $x_{N+1}-$axis and with $\|g\|_{p,2+\beta} < \infty$, equation (4.1) has a solution h satisfying the estimate*

$$\|h\|_{2,p,2+\beta} := \|D^2_\Sigma\, h\|_{p,2+\beta} \,+\, \|(1+r)^{1+\beta}\nabla_\Sigma\, h\|_{L^\infty(\Sigma)} \,+\, \|(1+r)^\beta h\|_{L^\infty(\Sigma)} \leq C\,\|g\|_{p,2+\beta}.$$

This solution is in addition axially symmetric and even respect to the $x_{N+1}-$ axis.

Proof. First recall that $\chi_\varepsilon : \mathbb{R}^N \to R$ is given by

$$\chi_\varepsilon(r) = \begin{cases} 0, & r > R_\varepsilon \\ 1, & r < R_\varepsilon \end{cases}$$

Since Σ coincides with the catenoid M for $1 < r < R_\varepsilon$ and with the graphs of $\pm F_\varepsilon$ for $r > R_\varepsilon$, it is natural to look for a solution h of the form

$$h = (1-\chi_\varepsilon)h_1 + \chi_\varepsilon h_2$$

under the constraints

$$h_1(R_\varepsilon) = h_2(R_\varepsilon), \quad \partial_r h_1(R_\varepsilon) = \partial_r h_2(R_\varepsilon) \tag{4.2}$$

so that h is C^1 over Σ.

We decompose g as well as

$$g = (1-\chi_\varepsilon)g + \chi_\varepsilon g =: g_1 + g_2$$

and we find that it suffices for h_1 and h_2 to solve the equations

$$\Delta_M h_1 + |A_M|^2 h_1 = g_1, \quad \text{in } M \tag{4.3}$$

$$\varepsilon^2 \left(\Delta_\Sigma h_2 + |A_\Sigma|^2 h_2\right) + 2\sqrt{2} a_0 \, e^{-2\sqrt{2}\varepsilon^{-1}F_\varepsilon} h_2 = \varepsilon^2 g_2, \quad \text{in } \Sigma \cap \{r > R_\varepsilon\} \tag{4.4}$$

with axially symmetric right hand sides g_i satisfying that $\|g_i\|_{p,2+\beta} < \infty$, $i = 1,2$ and under the constraints (4.2).

We begin by studying equation (4.3). Let us remark that the catenoid M can also be parameterized using the local coordinate system

$$(\mathrm{s}, \Theta) \in \mathbb{R} \times S^N \mapsto (\phi(\mathrm{s})\Theta, \psi(\mathrm{s})) \in M$$

where the function $\phi(\mathrm{s})$ satisfies the IVP

$$\dot{\phi}^2 + \phi^{4-2N} = \phi^2, \quad \mathrm{s} \in \mathbb{R}, \quad \phi(0) = 1, \quad \dot{\phi}(0) = 0$$

and $\psi(s)$ is given by the rule

$$\dot{\psi} = \phi^{2-N}, \quad \psi(0) = 0.$$

From this we find that ϕ is even and positive and ψ is an odd smooth and increasing diffeomorphism from $\mathbb{R}$ to $(-T, T)$ where we recall that

$$T = \int_1^\infty \frac{1}{\sqrt{s^{2(N-1)} - 1}} \, ds.$$

Furthermore, there exists a constant $a > 0$ such that

$$\phi(\mathrm{s}) = a \, e^{|\mathrm{s}|}(1 + \mathcal{O}(e^{-2(N-1)|\mathrm{s}|})), \quad \text{as } |\mathrm{s}| \to \infty. \tag{4.5}$$

Through the change of variables $r = \phi(\mathrm{s})$ and setting

$$h_1(r) = \phi^{\frac{2-N}{2}} v(\mathrm{s}), \quad g_1(r) = \phi^{-\frac{2+N}{2}} \, \hat{g}(s) \tag{4.6}$$

we find the conjugate form of equation (4.3), namely

$$\partial_{\mathrm{ss}} v(\mathrm{s}) - \left[\left(\frac{N-2}{2} \right)^2 - \frac{N(3N-2)}{4} \phi^{2-2N} \right] v(\mathrm{s}) = \hat{g}(\mathrm{s}), \quad \mathrm{s} \in \mathbb{R}. \tag{4.7}$$

Since we are in the axially symmetric even setting, we consider equation (4.7) with the initial condition $\partial_{\mathrm{s}} v(0) = 0$.

We know that the kernel associated to (4.7) contains at least the jacobi fields coming from dilations and translations along the coordinate axis, i.e. when looked through local coordinates, the functions

$$y \cdot \nu(y), \quad \nu(y) \cdot e_j, \quad j = 1, \ldots, N+1, \quad y \in M$$

are elements of the kernel of equation (4.7). We remark also that jacobi field coming from rotations around the axis of symmetry is zero.

We readily check that

$$z_1(y) = \nu(y) \cdot e_{N+1}, \quad z_2(y) = y \cdot \nu(y), \quad y \in M$$

are the only axially symmetric ones and in terms of the functions $\phi(\mathrm{s})$ and $\psi(\mathrm{s})$ we have that

$$z_1(s) = \phi^{\frac{N-4}{2}} \partial_{\mathrm{s}} \phi, \quad z_2(\mathrm{s}) = \phi^{\frac{N-4}{2}} (\partial_{\mathrm{s}} \phi \psi - T \partial_{\mathrm{s}} \psi \phi)$$

from where it directly follows that z_1 is odd and z_2 is even. Also from the asymptotics in (4.5) it follows that

$$z_1(\mathrm{s}) \sim e^{\frac{N-2}{2}\mathrm{s}}, \quad z_2(\mathrm{s}) \sim e^{-\frac{N-2}{2}\mathrm{s}}, \quad \mathrm{s} \to \infty$$

with wronskian equals 1.

Hence, using the variation of parameter formula we set

$$v(\mathrm{s}) = -z_1(\mathrm{s})\int_0^{\mathrm{s}} z_2(\varsigma)\hat{g}(\varsigma)d\varsigma + z_2(\mathrm{s})\int_0^{\mathrm{s}} z_1(\varsigma)\hat{g}(\varsigma)d\varsigma. \tag{4.8}$$

Next, we estimate the size of v. For any $\mathrm{s}_0 > 0$, $r_0 = \phi(\mathrm{s}_0) > 1$, and for any $j_0 \in \mathbb{N}$ such that $j_0 \le r_0 \le j_0 + 1$, we estimate

$$\begin{aligned}
\left|\int_0^{\mathrm{s}_0} z_2(\varsigma)\hat{g}(\varsigma)d\varsigma\right| &\le C\int_0^{\mathrm{s}_0} e^{-\frac{N-2}{2}\varsigma}\, e^{\frac{N+2}{2}\varsigma}|g_1(\phi(\varsigma))|d\varsigma\\
&\le C\int_1^{r_0} \xi^{N-1}\,\xi^{-(N-2)}|g_1(\xi)|d\xi\\
&\le C\sum_{j=1}^{j_0}\int_j^{j+1}\xi^{N-1}\,\xi^{-(N-2)}\,|g_1(\xi)|d\xi \le C\sum_{j=1}^{j_0} j^{-(N-2)}\int_j^{j+1}\xi^{N-1}|g_1(\xi)|d\xi\\
&\le C\sum_{j=1}^{j_0} j^{-(N-2)-2-\beta+(N-1)\frac{1}{p'}}\,j^{2+\beta}\|g_1\|_{L^p(S_\Sigma(Y(j,\theta);1))}\\
&\le C\,\|g_1\|_{p,2+\beta}\sum_{j=1}^{j_0} j^{-(N-1)\frac{1}{p}-1-\beta} \le C\|g_1\|_{p,2+\beta}\,(1+r_0)^{-(N-1)\frac{1}{p}-\beta}\\
&\le C\,\|g_1\|_{p,2+\beta}\,e^{[-(N-1)\frac{1}{p}-\beta]\mathrm{s}_0}.
\end{aligned}$$

Proceeding in the same fashion we estimate the second integral in (4.8) to obtain that

$$\begin{aligned}
\left|\int_0^{\mathrm{s}_0} z_1(\varsigma)\hat{g}(\varsigma)d\varsigma\right| &\le C\int_0^{\mathrm{s}_0} e^{-\frac{N-2}{2}\varsigma}\,e^{\frac{N+2}{2}\varsigma}|g_1(\phi(\varsigma))|d\varsigma \le C\int_0^{\mathrm{s}_0} e^{(N-1)\varsigma}|g_1(\phi(\varsigma))|e^{\varsigma}d\varsigma\\
&\le C\int_1^{r_0}\xi^{N-1}|g_1(\xi)|d\xi \le C\,\|g_1\|_{p,2+\beta}\sum_{j=1}^{j_0} j^{(N-1)\frac{1}{p'}-2-\beta}\\
&\le C\|g_1\|_{p,2+\beta}\,r_0^{-(N-1)\frac{1}{p}-\beta+N-2} \approx C\|g_1\|_{p,2+\beta}\,e^{[-(N-1)\frac{1}{p}-\beta+N-2]\mathrm{s}_0}.
\end{aligned}$$

From (4.6) and the previous estimates, it is direct to check that

$$\|(1+r(y)^{1+\beta})D_M\,h_1\|_{L^\infty(M)} + \|(1+r(y)^{\beta})h_1\|_{L^\infty(M)} \le C\|g_1\|_{p,2+\beta}$$

Using elliptic estimates and directly from equation (4.3) we obtain for every $p > 1$ and $\beta > 0$ that

$$\|h_1\|_{2,p,2+\beta} \le C\|g_1\|_{2+\beta}. \tag{4.9}$$

Next, we study equation (4.4)

$$\varepsilon^2\left(\Delta_\Sigma\,h_2 + |A_\Sigma|^2\,h_2\right) + 2\sqrt{2}a_0\,e^{-2\sqrt{2}\varepsilon^{-1}F_\varepsilon}\,h_2 = g_2, \quad r > R_\varepsilon$$

with the initial conditions

$$h_2(R_\varepsilon) = h_1(R_\varepsilon), \quad \partial_r h_2(R_\varepsilon) = \partial_r h_1(R_\varepsilon)$$

which can be estimated from (4.9) to obtain that

$$|R_\varepsilon^{1+\beta}\,\partial_r h_1(R_\varepsilon)| + |R_\varepsilon^{\beta}\,h_1(R_\varepsilon)| \le C\|g_1\|_{p,2+\beta}.$$

Expression (3.15) implies that for $r > R_\varepsilon$

$$\begin{aligned} 2\sqrt{2}a_0 e^{-2\sqrt{2}\varepsilon^{-1}F_\varepsilon} &= \frac{2(N-2)\,\varepsilon^2}{r^2}\, e^{\bar{\omega}_\varepsilon} \\ &= \frac{2(N-2)\,\varepsilon^2}{r^2} + \varepsilon^2 \mathcal{O}(r^{-2}\bar{\omega}_\varepsilon). \end{aligned}$$

Observe also that (3.15) together with (3.20) and (3.21) imply that

$$\Delta_\Sigma = \Delta_{\mathbb{R}^N} + \varepsilon^2 \mathcal{O}(r^{-2})\partial_{rr} + \varepsilon \mathcal{O}(r^{-1})\partial_r$$

and

$$c\,\varepsilon^2(1+r)^{-4} \le |A_\Sigma|^2 \le C\,\varepsilon^2(1+r)^{-4}, \quad r > R_\varepsilon.$$

Therefore, the model linear equation associated to (4.4) is

$$\Delta_{\mathbb{R}^N} \hat{h} + \frac{2(N-2)}{r^2}\,(1+p(r))\;\hat{h} = \hat{g}, \quad r > R_\varepsilon$$

where

$$p(r) = \frac{2(N-2)}{r^2}\left(e^{-\bar{\omega}_\varepsilon} - 1\right) \approx 2(N-2)r^{-2}\,\bar{\omega}_\varepsilon, \quad \text{as } r \to \infty$$

and whose linear differential operator resembles a Hardy operator at infinity.

Abusing the notation and making the scaling

$$\hat{h}(r) \sim \hat{h}\left(\frac{r}{R_\varepsilon}\right)$$

we find that it suffices to solve the problem

$$\Delta_{\mathbb{R}^N} \hat{h} + \frac{2(N-2)}{r^2}(1+p(r))\hat{h} = R_\varepsilon^2\,\hat{g}(r), \quad r > 1 \tag{4.10}$$

$$\hat{h}(1) = h_1(R_\varepsilon), \quad \partial_r \hat{h}(1) = R_\varepsilon \partial_r h_1(R_\varepsilon) \tag{4.11}$$

where

$$|\partial_r \hat{h}(1)| \,+\, |\hat{h}(1)| \le C\,R_\varepsilon^{-\beta}\,\|g_1\|_{p,2+\beta}.$$

From the classic theory in asymptotic integration of second order linear ODE, described for instance in chapter 11 in [22] or following the same lines of the proof of lemma 4.1 in [1], we find two smooth linearly independent elements of the kernel for (4.10) $z_\pm(r)$, such that in the radial variable they have the asymptotics at infinity

$$z_+(r) \approx r^{-\frac{N-2}{2}} \begin{cases} \cos\left(\sqrt{(N-2)(10-N)}\log(r)\right), & 3 \le N \le 9 \\ 1, \quad N = 10 \\ r^{+\frac{1}{2}\sqrt{(N-2)(N-10)}}, \quad N \ge 11 \end{cases}$$

$$z_-(r) \approx r^{-\frac{N-2}{2}} \begin{cases} \sin\left(\sqrt{(N-2)(10-N)}\log(r)\right), & 3 \le N \le 9 \\ \log(r), \quad N = 10 \\ r^{-\frac{1}{2}\sqrt{(N-2)(N-10)}}, \quad N \ge 11 \end{cases}$$

and these relations can be differentiated. Also we can compute the wronskian $W(r) = W(z_+, z_-)$ to find the asymptotics

$$r^{N-1}\mathrm{W}(r) = \begin{cases} -\sqrt{(N-2)(10-N)}, & 3 \le N \le 9 \\ -1, & N = 10 \\ -\sqrt{(N-2)(N-10)}, & N \ge 11. \end{cases} \tag{4.12}$$

Then we set as solution the function

$$\hat{h}(r) = A\, z_+(r) + B z_-(r) + \hat{z}(r) \tag{4.13}$$

where $\hat{z}(r)$ is the particular solution

$$\hat{z}(r) = -z_+(r)\int_1^r \varsigma^{N-1}\, z_-(\varsigma)\hat{g}(\varsigma)d\varsigma + z_-(r)\int_1^r \varsigma^{N-1}\, z_+(\varsigma)\hat{g}(\varsigma)d\varsigma.$$

Since

$$\hat{h}(1) = Az_+(1) + Bz_-(1)$$
$$\partial_r\hat{h}(1) = A\,\partial_r z_+(1) + B\,\partial_r z_-(1)$$

we find that

$$\begin{pmatrix} A \\ B \end{pmatrix} = W(1)^{-1}\begin{pmatrix} z_+(1) & z_-(1) \\ \partial_r z_+(1) & \partial_r z_-(1) \end{pmatrix}\begin{pmatrix} \hat{h}(1) \\ \partial_r\hat{h}(1) \end{pmatrix}$$

and using (4.11) and (4.12) we find that

$$|A| \,+\, |B| \le C\, R_\varepsilon^{-\beta}\, \|g_1\|_{p,2+\beta}.$$

In order to estimate the function $\hat{z}(r)$ we proceed in the exact same fashion as above to find that

$$\|\hat{z}\|_{2,p,2+\beta} \le C\, R_\varepsilon^{-\beta}\|g_2\|_{p,2+\beta}$$

Rescaling back we obtain that

$$h_2(r) = Az_+\left(\frac{r}{R_\varepsilon}\right) + Bz_-\left(\frac{r}{R_\varepsilon}\right) + \hat{z}\left(\frac{r}{R_\varepsilon}\right), \quad r > R_\varepsilon$$

from where we find that

$$\|h_2\|_{2,p,2+\beta} \le C\|g_2\|_{p,2+\beta}$$

provided that $0 < \beta < \frac{N-2}{2}$ which is a slower decay than the one of the functions $z_\pm(r)$.

To finish the proof, let us denote by $\mathcal{L}(g_2) := h_2$ the resolvent operator for the model linear equation (4.10) . We apply a fixed point argument to the expression

$$\mathcal{L}\left[g_2 - |A_\Sigma|^2 h_2 + (\Delta_\Sigma - \Delta_{\mathbb{R}^N})\, h_2\right] = h_2$$

in the topologies described above, so that for $\varepsilon > 0$ small enough we find a unique solution to the equation satisfying the same estimate and this completes the proof of the Proposition. □

5. Approximation and preliminary discussion

Now we are in position to describe the approximation we want to consider. Denote by $w(t)$ the heteroclinic solution to the one dimensional Allen-Cahn equation

$$w'' + w(1-w^2) = 0, \quad \text{in } \mathbb{R}, \quad w(\pm\infty) = \pm 1, \quad w'(t) > 0 \tag{5.1}$$

which is given explicitely by

$$w(t) = \tanh\left(\frac{t}{\sqrt{2}}\right), \quad t \in \mathbb{R}$$

and has the following asymptotic behavior

$$w(t) = \begin{cases} 1 - 2\, e^{-\sqrt{2}\, t} + \mathcal{O}(e^{-2\sqrt{2}\, t}), & t > 0 \\ \\ -1 + 2\, e^{\sqrt{2}\, t} + \mathcal{O}(e^{2\sqrt{2}\, t}), & t < 0 \end{cases} \tag{5.2}$$

and relation (5.2) can be differentiated so that

$$w'(t) = 2\sqrt{2}e^{-\sqrt{2}|t|} + \mathcal{O}(e^{-2\sqrt{2}\,|t|}), \quad t \in \mathbb{R}.$$

$$w''(t) = -4\, e^{-\sqrt{2}|t|} + \mathcal{O}(e^{-2\sqrt{2}\,|t|}), \quad t \in \mathbb{R}.$$

From (3.22) we consider the set $\mathcal{N}_\varepsilon = \mathcal{N}_{\varepsilon,+} \cup \mathcal{N}_{\varepsilon,-}$, the cylinder

$$C_{R_\varepsilon} := \left\{ y \in \varepsilon^{-1}\,\Sigma : r(\varepsilon y) < R_\varepsilon \right\}.$$

and the cut-off function $\eta_\varepsilon(r) := \eta(\varepsilon r - R_\varepsilon)$ with

$$\eta(\mathrm{s}) = \begin{cases} 0, & \mathrm{s} < -1 \\ 1, & \mathrm{s} > 1. \end{cases}$$

Recall also that in sections 2 and 3 we introduced a smooth bounded parameter function h satisfying the apriori estimate (3.23).

We define our approximation to (1.1) in the region $\mathcal{N}_\varepsilon$ as the function

$$\bar{u}(x) = (1 - \eta_\varepsilon)u_0 + \eta_\varepsilon u_1 \tag{5.3}$$

where

$$u_0(x) = w(z - h(\varepsilon r)), \quad x \in \mathcal{N}_\varepsilon \cap C_{R_\varepsilon} \tag{5.4}$$

and taking into account (3.26)

$$\begin{aligned} u_1(x) = {} & w(\sqrt{1 + |\partial_r F_\varepsilon(\varepsilon r)|^2}\,\big(x_{N+1} - \varepsilon^{-1} F_\varepsilon(\varepsilon r)\big) - h(\varepsilon r)) \\ & - w(\sqrt{1 + |\partial_r F_\varepsilon(\varepsilon r)|^2}\,\big(x_{N+1} + \varepsilon^{-1} F_\varepsilon(\varepsilon r)\big) + h(\varepsilon r)) - 1, \quad x \in \mathcal{N}_\varepsilon - C_{R_\varepsilon}. \end{aligned} \tag{5.5}$$

Observe that in $\mathcal{N}_\varepsilon \cap C_{R_\varepsilon}$ the function u_0 coincides with the heteroclinic solution while, from (3.26)

$$u_1(x) \;\approx\; w(x_{N+1} - \varepsilon^{-1} F_\varepsilon(\varepsilon r) - h(\varepsilon r)) \;-\; w(z + \varepsilon^{-1} F_\varepsilon(\varepsilon r) + h(\varepsilon r)) - 1$$

for $x = y + z\nu_\varepsilon(y) \in \mathcal{N}_\varepsilon$ with $r(\varepsilon y) > R_\varepsilon$.

Since $\mathbb{R}^{N+1} - \Sigma = S_+ \cup S_-$, we also have the associated dilated version $\mathbb{R}^{N+1} - \Sigma_\varepsilon = S_{\varepsilon,+} \cup S_{\varepsilon,-}$ where $S_{\varepsilon,\pm} := \varepsilon^{-1} S_\pm$ respectively. Consider the function $\mathbb{H}$ given by

$$\mathbb{H}(x) = \begin{cases} 1, & \text{in } S_{\varepsilon,+} \\ -1, & \text{in } S_{\varepsilon,-}. \end{cases}$$

As global approximation we take the interpolation between the functions $\bar{u}$ and $\mathbb{H}$. To be precise we use a cut-off function ρ_ε described by

$$\rho_\varepsilon(x) = \eta\left(|z| - \frac{\delta_0}{\varepsilon}(1 - \chi_\varepsilon) + \chi_\varepsilon \varepsilon^{-1} F_\varepsilon(\varepsilon r) + 1 \right), \quad x = X_\varepsilon(r, \Theta, z) \in \mathcal{N}_\varepsilon$$

to write

$$\mathrm{w}_\varepsilon(x) = \rho_\varepsilon \bar{u}(x) + (1 - \rho_\varepsilon)\mathbb{H}(x), \quad \text{in } \mathbb{R}^{N+1}. \tag{5.6}$$

Our next and crucial step is to compute the error of this approximation. Let us write

$$S(\mathrm{w}_\varepsilon) := \Delta \mathrm{w}_\varepsilon + F(\mathrm{w}_\varepsilon)$$

where

$$F(\mathrm{w}_\varepsilon) := \mathrm{w}_\varepsilon(1 - \mathrm{w}_\varepsilon^2).$$

Observe that $S(\mathrm{w}_\varepsilon)$ splits into

$$S(\mathrm{w}_\varepsilon) \;=\; \underbrace{\rho_\varepsilon(x) S(\bar{u})}_{I} \;+\; \underbrace{2\rho_\varepsilon \cdot \nabla(\bar{u} - \mathbb{H}) + (\bar{u} - \mathbb{H})\Delta\rho_\varepsilon}_{II} \;+\; \underbrace{F(\mathrm{w}_\varepsilon) - \rho_\varepsilon F(\bar{u})}_{III} \tag{5.7}$$

so that we compute the sizes of each of the terms involved in expression (5.7).

First we compute I. Using the characteristic function χ_ε as in (3.17)

$$S(\bar u) = (1-\chi_\varepsilon(\varepsilon r))S(\bar u) + \chi_\varepsilon(\varepsilon r)S(\bar u). \tag{5.8}$$

Setting $z = t + h(\varepsilon r)$, we find from expression (3.24) that in the region $\mathcal{N}_{\varepsilon,h}\cap C_{R_\varepsilon}$, $\bar u(x) = w(t)$. So that

$$\begin{aligned} S(\bar u(x)) &= -\varepsilon H_\Sigma(\varepsilon r) w'(t) - \varepsilon^2 |A_\Sigma(\varepsilon r)|^2 t\, w'(t) \\ &-\varepsilon^2 \left\{\Delta_\Sigma h \,+\, |A_\Sigma|^2 h\right\} w'(t) - \varepsilon^2 \left[\partial_r h(\varepsilon r)\right]^2 w''(t) \\ &-\varepsilon^3 (t+h) a_1(\varepsilon r, \varepsilon(t+h)) \left(\partial_{rr} h(\varepsilon r) w'(t) + [\partial_r h(\varepsilon r)]^2 w''(t)\right) \\ &- \varepsilon^3 (t+h) b_1(\varepsilon r, \varepsilon(t+h))\, \partial_r h(\varepsilon r) w'(t) + \varepsilon^3 (t+h)^2 b_{N+1}(\varepsilon r, \varepsilon(t+h))\, w'(t). \end{aligned} \tag{5.9}$$

The minimality of M implies that $H_\Sigma = H_M = 0$ in the cylinder C_{R_ε} and we conclude that

$$|S(\bar u(x)) + \mathcal{J}_\Sigma(h)(\varepsilon r) w'(t)| \le C\varepsilon^2 (1+\varepsilon r)^{-2} e^{-\sigma|t|}$$

for any $0 < \sigma < \sqrt{2}$.

As for the second term in (5.8), setting

$$\begin{aligned} t &= \sqrt{1+|\partial_r G(\varepsilon r)|^2}\left(x_{N+1} - \varepsilon^{-1}\varepsilon^{-1} G_\varepsilon(\varepsilon r) - h(\varepsilon r)\right) \\ &= z - h(\varepsilon r) \end{aligned}$$

in the set $\Sigma_{\varepsilon,+}$ for $r > R_\varepsilon$, we observe that

$$\bar u(x) = w(t) - w(t + 2\varepsilon^{-1} F_\varepsilon(\varepsilon r) + 2h(\varepsilon r)) - 1$$

and consequently

$$\begin{aligned} F(\bar u) &= F\big(w(t)\big) \,-\, F\big(w(t + 2\varepsilon^{-1} F_\varepsilon(\varepsilon r) + 2h(\varepsilon r))\big) - 1 \,+ 1 \\ &- \left(F'(w(t)) - F'(-1)\right)\left[w(t + 2\varepsilon^{-1} F_\varepsilon + 2h(\varepsilon r)) + 1\right] \\ &+ \frac{1}{2}\left(F''(w(t)) +\, F'(-1)\right)\left[w(t + 2\varepsilon^{-1} F_\varepsilon + 2h(\varepsilon r)) + 1\right]^2 \,+\, \mathcal{O}\left(\left[w(t + 2F_\varepsilon + 2h(\varepsilon r)) + 1\right]^3\right). \end{aligned}$$

From the asymptotics in (5.2) and since h is even in Σ, we obtain that

$$\begin{aligned} F(\bar u) \;&=\; F(w(t)) - F(w(t + 2\varepsilon_\varepsilon^{-1} + 2h)) - 6(1-w^2(t)) e^{\sqrt{2}t} e^{-2\sqrt{2}(\varepsilon^{-1}F_\varepsilon + h)} \\ &+ 6\left[(1-w^2(t)) + 2(1-w(t))\right] e^{2\sqrt{2}t} e^{-2\sqrt{2}(\varepsilon^{-1}F_\varepsilon + h)} \,+\, \mathcal{O}\left(e^{-3\sqrt{2}|t + 2\varepsilon^{-1}F_\varepsilon + 2h|}\right). \end{aligned} \tag{5.10}$$

Let us now consider the decomposition

$$6(1-w^2(t)) e^{-\sqrt{2}t} = a_0\, w'(t) \,+\, g_0(t), \qquad \int_{\mathbb{R}} g_0(t) w'(t) dt \,=0. \tag{5.11}$$

Using the function g_0 and (5.10) we find that in $\Sigma_{\varepsilon,+}$ and for $r > R_\varepsilon$

$$\begin{aligned} S(\bar u) \;&=\; -\left(\varepsilon H_\Sigma(\varepsilon r) - a_0\, e^{-2\sqrt{2}\,\varepsilon^{-1}F_\varepsilon}\right) w'(t) \,-\, \varepsilon^2 |A_\Sigma|^2 t\, w'(t) \,+\, g_0(t) e^{-2\sqrt{2}\varepsilon^{-1}F_\varepsilon} \\ &- \varepsilon^2\left(\Delta_\Sigma h \,+ |A_\Sigma|^2 h\right) w'(t) \,-\, 6\left(1-w^2(t)\right) e^{\sqrt{2}t} e^{-2\sqrt{2}\varepsilon^{-1}F_\varepsilon}\left[e^{-2\sqrt{2}h} \,-\, 1\right] \\ &+ \varepsilon^2 |\nabla_\Sigma h|^2\, w''(t) \,+\, \varepsilon^2\left(\Delta_\Sigma h + |A_\Sigma|^2 h\right) w'(t + 2\varepsilon^{-1}F_\varepsilon + 2h) \,+\, \varepsilon^2 |\nabla_\Sigma h|^2 w''(t - 2\mathrm{f}_\alpha) \\ &+\, \left[6(1-w^2(t)) + 12(1-w(t))\right] e^{2\sqrt{2}t} e^{-4\sqrt{2}(\varepsilon^{-1}F_\varepsilon + h)} \,+\, \widetilde{\mathrm{R}} \end{aligned} \tag{5.12}$$

and

$$|D_p\widetilde{\mathrm{R}}(\varepsilon r,t,p,q)| \,+\, |D_q\widetilde{\mathrm{R}}(\varepsilon r,t,p,q)| \,+\, |\widetilde{\mathrm{R}}(\varepsilon r',t,p,q)| \leq C\varepsilon^{2+\tau}(1+|\varepsilon r|)^{-4}e^{-\sigma|t|} \tag{5.13}$$

for any $0<\sigma<\sqrt{2}$ and any $0<\tau<1$. Similar computations are obtained in the set $\Sigma_{\varepsilon,-}$ for $r>R_\varepsilon$.

The asymptotics for F_ε in (3.15) imply that for $r>R_\varepsilon$

$$H_\Sigma \,=\, \nabla\cdot\left(\frac{\nabla F_\varepsilon}{\sqrt{1+|\nabla F_\varepsilon|^2}}\right) \,=\, \Delta_{\mathbb{R}^N}F_\varepsilon \,+\, \varepsilon^3\,\mathcal{O}((1+r)^{-4}).$$

Using the fact that the function F_ε is an exact solution of equation (3.3) we can write

$$E_0(\varepsilon r) \,:=\, \varepsilon\,H_\Sigma(\varepsilon r) - a_0\,e^{-2\sqrt{2}\,\varepsilon^{-1}F_\varepsilon} = \varepsilon^4\mathcal{O}((1+\varepsilon r)^{-4}) \tag{5.14}$$

and from (5.14) we observe that for $\varepsilon r>R_\varepsilon$

$$\begin{aligned} S(\bar u) \,=\, &-\varepsilon^2\left(\Delta_\Sigma h\,+|A_\Sigma|^2\,h\right)w'(t) \,-\, 6\left(1-w^2(t)\right)e^{\sqrt{2}t}\,e^{-2\sqrt{2}\varepsilon^{-1}F_\varepsilon}\left[e^{-2\sqrt{2}h}\,-\,1\right] \\ &-E_0(\varepsilon r)w'(t) \,-\, \varepsilon^2|A_\Sigma|^2 t\,w'(t) \,+\, g_0(t)e^{-2\sqrt{2}\varepsilon^{-1}F_\varepsilon} \,+\, \varepsilon^2\,|\nabla_\Sigma h|^2\,w''(t) \\ &+\,\varepsilon^2\,(\Delta_\Sigma h+|A_\Sigma|^2\,h)\,w'(t+2\varepsilon^{-1}F_\varepsilon+2h) \,+\, \varepsilon^2|\nabla_\Sigma h|^2 w''(t-2\mathrm{f}_\alpha) \,+\, \widetilde{\mathrm{R}} \end{aligned} \tag{5.15}$$

where $\widetilde{\mathrm{R}}$ is as described in (5.13).

From this we find that for $\varepsilon r>R_\varepsilon$

$$|S(\bar u)\chi_\varepsilon| \leq C\varepsilon^{2-\tau}(1+\varepsilon r)^{-2+\tau}e^{-\sigma\,|t|}.$$

From (5.9) and (5.15) that for any $0<\sigma<\sqrt{2}$ and any $\tau=\frac{\sigma}{\sqrt{2}}$

$$\|S(\bar u)\|_{p,2-\tau,\sigma} \leq C\varepsilon^{2-\tau}.$$

Since the cut-off function ρ_ε is supported in a region of the form

$$\frac{\delta_0}{\varepsilon}(1-\chi_\varepsilon)+\chi_\varepsilon\varepsilon^{-1}F_\varepsilon(\varepsilon r)-1 \leq |z| \leq \frac{\delta_0}{\varepsilon}(1-\chi_\varepsilon)+\chi_\varepsilon\varepsilon^{-1}F_\varepsilon(\varepsilon r)+1$$

the contribution of the term II in (5.7) is

$$|2\nabla\rho_\varepsilon\cdot\nabla\bar u(x)+(\bar u(x)-\mathbb{H}(x))\Delta\rho_\varepsilon| \leq C\mathrm{e}^{\sqrt{2}|t+2\chi_\varepsilon\varepsilon^{-1}F_\varepsilon+2h|}.$$

As for the term II in (5.7), we can easily check that

$$III = 3(\bar u-\mathbb{H})^2\rho_\varepsilon(1-\rho_\varepsilon)+3(\bar u-\mathbb{H})^3\rho_\varepsilon(1-\rho_\varepsilon^3)$$

so that

$$|II|+|III| \leq Ce^{-\sqrt{2}|t|} \leq Ce^{-\sigma|t|} \leq C\varepsilon^{2-\frac{\sigma}{\sqrt{2}}}(1+\varepsilon r)^{-2+\frac{\sigma}{\sqrt{2}}}e^{-\sigma|t|}.$$

Putting together the estimates for I, II, III, we find that

$$\|S(\mathrm{w}_\varepsilon)\|_{p,2-\tau,\sigma} \leq C\varepsilon^{2-\tau} \tag{5.16}$$

for any $\sigma\in(0,\sqrt{2})$ and any $\tau=\frac{\sigma}{\sqrt{2}}$.

We remark also that in the upper part of the set $\mathcal{N}_{\varepsilon,h}$

$$\begin{aligned} S(\bar u) \,=\, &-\varepsilon^2\left\{\Delta_\Sigma h\,+|A_\Sigma|^2\,h\right\}w'(t) \,-\, 6\left(1-w^2(t)\right)e^{\sqrt{2}t}\,\chi_\varepsilon\,2\sqrt{2}\,e^{-2\sqrt{2}\varepsilon^{-1}F_\varepsilon}\,h \\ &-\chi_\varepsilon\,E_0(\varepsilon r)w'(t) \,+\, g_0(t)e^{-2\sqrt{2}\varepsilon^{-1}F_\varepsilon} \,-\, \varepsilon^2|A_\Sigma|^2 t\,w'(t) \,+\, \varepsilon^2\,|\nabla_\Sigma h|^2\,w''(t) \end{aligned}$$

$$-6(1-w^2(t))e^{\sqrt{2}t}\chi_\varepsilon(\varepsilon r)\, e^{-2\sqrt{2}\varepsilon^{-1}F_\varepsilon}\left[e^{-2\sqrt{2}h}\,-\,1+2\sqrt{2}\,h\right]+R.$$

and where

$$|D_pR(\varepsilon r,t,p,q)|\,+\,|D_qR(\varepsilon r,t,p,q)|\,+\,|R(\varepsilon r',t,p,q)|\le C\varepsilon^{2-\tau}(1+|\varepsilon r|)^{-2+\tau}e^{-\sigma|t|} \tag{5.17}$$

for some $0<\sigma<\sqrt{2}$ and some $\tau=\frac{\sigma}{\sqrt{2}}$.

6. Lyapunov reduction Scheme

We begin this section by setting up the functional analytic spaces we need to carry out the proof of Theorem 1.

Let us first consider the function $r: R^{N+1}\to[0,\infty)$ given by

$$r(x',x_{N+1}):=|y'|,\quad x=(x',x_{N+1})\in\mathbb{R}^{N+1}$$

Let us define for $\varepsilon>0$, $\mu>0$ and $f(x)$, defined in $\mathbb{R}^{N+1}$, the norm

$$\|f\|_{p,\mu,\sim}\,:=\sup_{x\in\mathbb{R}^{N+1}}(1+r(\varepsilon x))^{\mu}\,\|f\|_{L^p(B_1(x))},\quad p>1. \tag{6.1}$$

We also consider $0<\sigma<\sqrt{2}$, $\mu>0$, $\varepsilon>0$ and functions $g=g(y,t)$ and $\phi=\phi(y,t)$, defined for every $(y,t)\in\Sigma_\varepsilon\times\mathbb{R}$. Let us set the norms

$$\|g\|_{p,\mu,\sigma}:=\sup_{(y,t)\in\Sigma_\varepsilon\times\mathbb{R}}(1+r(\varepsilon y))^{\mu}e^{\sigma\,|t|}\|g\|_{L^p(B_1(y,t))} \tag{6.2}$$

$$\|\phi\|_{\infty,\mu,\sigma}:=\|(1+r(\varepsilon y)^{\mu})e^{\sigma\,|t|}\phi\|_{L^\infty(\Sigma_\varepsilon\times\mathbb{R})} \tag{6.3}$$

$$\|\phi\|_{2,p,\mu,\sigma}:=\|D^2\,\phi\|_{p,\mu,\sigma}\,+\,\|D\phi\|_{\infty,\mu,\sigma}\,+\,\|\phi\|_{\infty,\mu,\sigma}. \tag{6.4}$$

Finally, for functions h and $\widetilde{g}$ defined in Σ, we recall the norms

$$\|\widetilde{g}\|_{p,\beta}:=\sup_{y\in\Sigma}(1+r(y))^{\beta}\,\|\widetilde{g}\|_{L^p(S_\Sigma(y;1))} \tag{6.5}$$

$$\|h\|_{2,p,\beta}:=\|D^2h\|_{p,\beta}\,+\,\|(1+r(y))\,D\,h\|_{L^\infty(\Sigma)}\,+\,\|h\|_{L^\infty(\Sigma)}. \tag{6.6}$$

Now, in order to prove Theorem 1, let us look for a solution to equation (1.1) of the form

$$u_\varepsilon(x)\,=\,\mathrm{w}_\varepsilon(x)+\varphi(x)$$

where $\mathrm{w}_\varepsilon(x)$ is the global approximation defined in (5.6) and φ is going to be chosen small in an appropriate topology.

Denoting $F(u)=u(1-u^2)$, we observe that for $u_\varepsilon(x)$ to be a solution to (1.1), the function φ must solve the equation

$$\Delta\varphi\,+\,F'(\mathrm{w}_\varepsilon)\varphi\,+\,S(\mathrm{w}_\varepsilon)+N(\varphi)\,=\,0,\quad\text{in }\mathbb{R}^{N+1}$$

or equivalently

$$\Delta\varphi\,+\,F'(\mathrm{w}_\varepsilon)\varphi\quad=\quad-S(\mathrm{w}_\varepsilon)-N(\varphi) \tag{6.7}$$

where

$$N(\varphi)\,=\,F(\mathrm{w}_\varepsilon+\varphi)-F(\mathrm{w}_\varepsilon)-F'(\mathrm{w}_\varepsilon).$$

6.1 Gluing procedure: The strategy here consists in transforming (6.7) into a system of two PDEs. One of these equations will take account of the geometry of Σ and w_ε near Σ_ε, while the other will handle the situation far away from Σ_ε, where w_ε is almost constant.

In order to proceed, we consider again the non-negative cut-off function $\zeta \in C^\infty(\mathbb{R})$ such that

$$\zeta(s) \,=\, \begin{cases} 1, & s<-1 \\ 0, & s>1 \end{cases}$$

and define for $n \in \mathbb{N}$, the cut off function for $x = X_{\varepsilon,\pm,h}(y,t) \in \mathcal{N}_{\varepsilon,\pm,h}$

$$\zeta_n(x) \,=\, \zeta\left(|t+h| - \frac{\delta_0}{\varepsilon}\,(1-\eta_\varepsilon) - \eta_\varepsilon \varepsilon^{-1} F_\varepsilon(\varepsilon r) + n \right) \tag{6.8}$$

for $x = X_{\varepsilon,\pm,h}(r,\Theta,t) \in \mathcal{N}_{\varepsilon,\pm,h}$.

Therefore, we look for a solution $\varphi(x)$ with the particular form

$$\varphi(x) \,=\, \zeta_3\,(x)\,\phi(y,t) \,+\, \psi(x)$$

where the function $\phi(y,t)$ is defined in $\Sigma_\varepsilon \times \mathbb{R}$ and the function $\psi(x)$ is defined in the whole $\mathbb{R}^{N+1}$.

From equation (6.7) and noticing that $\zeta_2 \cdot \zeta_3 = \zeta_3$, we find that

$$\zeta_3\left[\Delta_{\mathcal{N}_{\varepsilon,h}}\phi + F'(\zeta_2 \mathrm{w}_\varepsilon)\phi + \zeta_2 S(\mathrm{w}_\varepsilon) + \zeta_2 N(\phi+\psi) + \zeta_2 (F'(\mathrm{w}_\varepsilon)+2)\psi\right]$$

$$+\,\Delta\psi - [2-(1-\zeta_3)(F'(\mathrm{w}_\varepsilon)+2)]\psi + (1-\zeta_3)\,S(\mathrm{w}_\varepsilon)$$

$$+\nabla\zeta_3 \cdot \nabla_{\mathcal{N}_{\varepsilon,h}}\phi + \phi\Delta\zeta_3 + (1-\zeta_3)N[\psi+\zeta_2\phi] = 0.$$

Hence, to solve (6.7), it suffices to solve the system of PDEs

$$\begin{aligned}\Delta\psi - [2-(1-\zeta_2)(F'(\mathrm{w}_\varepsilon)+2)]\,\psi = &-\,(1-\zeta_2)\,S(\mathrm{w}_\varepsilon) - \\ &-\,2\,\nabla\zeta_2\cdot\nabla_{\mathcal{N}_{\varepsilon,h}}\phi \,-\, \phi\Delta\zeta_2 \,-\, (1-\zeta_3)\,N\left[\zeta_2\phi+\psi\right], \quad \text{in } \mathbb{R}^{N+1}\end{aligned} \tag{6.9}$$

$$\begin{aligned}\Delta_{\mathcal{N}_{\varepsilon,h}}\phi + F'(\zeta_2 \mathrm{w}_\varepsilon)\phi = &-\zeta_2 S(\mathrm{w}_\varepsilon) \,-\, \zeta_2 N(\phi+\psi) \\ &-\,\zeta_2(F'(\mathrm{w}_\varepsilon)+2)\psi, \quad \text{for } |t+h| \le \varrho_\varepsilon(r)\end{aligned} \tag{6.10}$$

where

$$\varrho_\varepsilon(y) := \frac{\delta_0}{\varepsilon}\,(1-\eta_\varepsilon(\varepsilon r)) + \eta_\varepsilon(\varepsilon r)\varepsilon^1 F_\varepsilon(\varepsilon r) - 2, \quad y = Y_\varepsilon(r,\Theta) \in \Sigma_\varepsilon$$

Now, we extend equation (6.10) to the whole $\Sigma_\varepsilon \times \mathbb{R}$. First, let us introduce the differential operator

$$B := \zeta_2[\Delta_{\mathcal{N}_{\varepsilon,h}} - \partial_{tt} - \Delta_{\Sigma_\varepsilon}].$$

Recall that $\Delta_{\Sigma_\varepsilon}$ is nothing but the Laplace-Beltrami operator of Σ_ε and which in the local coordinates $Y_{\varepsilon,\pm}(r,\Theta)$, has the expression

$$\Delta_{\Sigma_{\varepsilon,\pm}} = \partial_{rr} + \frac{N-1}{r}\partial_r + \frac{1}{r^2}\Delta_{S^{N-1}} + \varepsilon^2\mathcal{O}(\varepsilon^{-2}r^{-2})\partial_r r + \varepsilon^3\mathcal{O}(\varepsilon^{-3}r^{-3})\partial_r$$

Clearly, B vanishes in the domain

$$|t+h| \ge \frac{\delta_0}{\varepsilon}\,(1-\eta_\varepsilon(\varepsilon r)) + \eta_\varepsilon(\varepsilon r)\varepsilon^1 F_\varepsilon(\varepsilon r) - 1, \quad y = Y_\varepsilon(r,\Theta) \in \Sigma_\varepsilon$$

and so, instead of equation (6.10), we consider the equation

$$\begin{aligned}\partial_{tt}\phi + \Delta_{\Sigma_\varepsilon}\phi + F'(w(t))\phi = &-\widetilde{S}(\mathrm{w}_\varepsilon) - B(\phi) \\ -\,[F'(\zeta_2\mathrm{w}_\varepsilon) - F'(w(t))]\phi \,-\, &\zeta_2(F'(\mathrm{w}_\varepsilon)+2)\psi \,-\, \zeta_2 N(\phi+\psi), \quad \text{in } M_\alpha \times \mathbb{R}\end{aligned} \tag{6.11}$$

and where we have denoted

$$\widetilde{S}(\mathrm{w}_\varepsilon) \,=\, -\,\varepsilon^2\left(\Delta_\Sigma h \,+ |A_\Sigma|^2\,h\right)w'(t) \,-\, 6\left(1-w^2(t)\right)e^{\sqrt{2}t}\,\chi_\varepsilon\,2\sqrt{2}\,e^{-2\sqrt{2}\varepsilon^{-1}F_\varepsilon}\,h$$

$$-\chi_\varepsilon\,E_0(\varepsilon r)w'(t) \,-\, \varepsilon^2|A_\Sigma|^2 t\,w'(t) \,+\, g_0(t)e^{-2\sqrt{2}\varepsilon^{-1}F_\varepsilon} \,+\, \varepsilon^2\,|\nabla_\Sigma h|^2\,w''(t)$$

$$-2\sqrt{2}c_0a_0\chi_\varepsilon(\varepsilon r)\, e^{-2\sqrt{2}\varepsilon^{-1}F_\varepsilon}\left[e^{-2\sqrt{2}h}\, -\, 1\, +2\sqrt{2}\, h\right]+\zeta_1\, R.$$

and where

$$|D_pR(\varepsilon r,t,p,q)|\, +\, |D_qR(\varepsilon r,t,p,q)|\, +\, |R(\varepsilon r',t,p,q)|\leq C\varepsilon^{2-\tau}(1+|\varepsilon r|)^{-2+\tau}e^{-\sigma|t|} \tag{6.12}$$

for any $0<\sigma<\sqrt{2}$ and $\tau=\frac{\sigma}{\sqrt{2}}$.

Observe that $\widetilde{S}(\mathrm{w}_\varepsilon)$ coincides with $S(\mathrm{w}_\varepsilon)$ where $\zeta_1=1$, but we have basically cut-off the parts in $S(\mathrm{w}_\varepsilon)$ that, in the local coordinates $X_{\varepsilon,\pm,h}$, are not defined for all $t\in\mathbb{R}$.

Using (5.16) and since the support of ζ_2 is contained in a region of the form

$$|t+h|\leq\frac{C}{\varepsilon}(1-\eta_\varepsilon)+\frac{\eta_\varepsilon}{2\sqrt{2}}\left(\log(\frac{2\sqrt{2}}{\varepsilon^2})+\log(\varepsilon^2r^2)+\log(2(N-2))\right)$$

we compute directly the size of this error to obtain that

$$\|\widetilde{S}(\mathrm{w}_\varepsilon)\|_{p,2-\frac{\sigma}{\sqrt{2}},\sigma}\leq C\varepsilon^{2-\frac{\sigma}{\sqrt{2}}} \tag{6.13}$$

for any $\sigma\in(0,\sqrt{2})$.

To solve system (6.9)-(6.11), we first solve equation (6.9), using the fact that the potential $2-(1-\zeta_3)(F'(\mathrm{w}_\varepsilon)+2)$ is uniformly positive, so that the linear operator there behaves like $\Delta_{\mathbb{R}^{N+1}}-2$. A solution $\psi=\Psi(\phi)$ is then found using contraction mapping principle. We collect this discussion in the following proposition, that will be proven in detail in section 7.

Proposition 6.1. *Assume* $0<\sigma<\sqrt{2}$, $\mu>0$, $p>2$ *and let the function* h *satisfy* (3.23). *Then, for every* $\varepsilon>0$ *sufficiently small and for a fixed function* ϕ, *satisfying that*

$$\|\phi\|_{2,p,\mu,\sigma}\leq 1$$

equation (6.9) *has a unique solution* $\psi=\Psi(\phi)$. *Even more, the operator* $\psi=\Psi(\phi)$ *turns out to be Lipschitz in* ϕ. *More precisely,* $\psi=\Psi(\phi)$ *satisfies that*

$$\begin{aligned}\|\psi\|_X \;&:=\; \|D^2\psi\|_{p,\hat{\mu},\sim}+\|(1+r^{\hat{\mu}}(\alpha x))\,D\psi\|_{L^\infty(\mathbb{R}^{N+1})}+\|(1+r^{\hat{\mu}}(\alpha x))\,\psi\|_{L^\infty(\mathbb{R}^{N+1})}\\ &\leq\; C\left(\alpha^{2+\frac{\sigma}{\sqrt{2}}-\alpha}+\varepsilon^{\frac{\sigma}{\sqrt{2}}-\alpha}\|\phi\|_{2,p,\mu,\sigma}\right)\end{aligned} \tag{6.14}$$

where $0<\hat{\mu}<\min(2\mu\,,\,\mu+\varrho\sqrt{2}\,,\,2+\varrho\sqrt{2})$ *and*

$$\|\Psi(\phi)-\Psi(\hat{\phi})\|_X\leq C\varepsilon^{\frac{\varrho}{\sqrt{2}}-\alpha}\|\phi-\hat{\phi}\|_{2,p,\mu,\sigma}. \tag{6.15}$$

Hence, using Proposition 6.1, we solve equation (6.11) with $\psi=\Psi(\phi)$. Let us set

$$\mathbf{N}(\phi)\,:=\,B(\phi)\,+\,[F'(\zeta_2\mathrm{w}_\varepsilon)-F'(w(t))]\phi\,+\,\zeta_2(F'(\mathrm{w}_\varepsilon)+2)\Psi(\phi)\,+\,\zeta_2N[\phi+\Psi(\phi)].$$

We need to solve

$$\partial_{tt}\phi\,+\,\Delta_{\Sigma_\varepsilon}\phi\,+\,F'(w(t))\,\phi\,=\,-\,\widetilde{S}(\mathrm{w}_\varepsilon)-\mathbf{N}(\Phi),\quad\text{in }\Sigma_\varepsilon\times\mathbb{R}. \tag{6.16}$$

To solve system (6.16), we solve a nonlinear and nonlocal problem for ϕ, in such a way that we eliminate the parts of the error that do not contribute to the projections onto $w'(t)$. This step can be though as an improvement of the approximation w_ε.

We use the fact that the error has the size

$$\|\widetilde{S}(\mathrm{w}_\varepsilon)\|_{p,2-\frac{\sigma}{\sqrt{2}},\sigma}\leq\varepsilon^{2\,-\,\frac{\sigma}{\sqrt{2}}} \tag{6.17}$$

and, as we will see in section 7, for $\tau = \frac{\sigma}{\sqrt{2}} > 0$, $\mathbf{N}(\phi)$ satisfies that

$$\|\mathbf{N}(\phi)\|_{p,4-\frac{\sigma}{\sqrt{2}},\sigma} \le C\varepsilon^{3-\tau} \tag{6.18}$$

$$\|\mathbf{N}(\phi_1) - \mathbf{N}(\phi_2)\|_{p,4-\frac{\sigma}{\sqrt{2}},\sigma} \le C\,\varepsilon\,\|\phi_1 - \phi_2\|_{2,p,2,\sigma}, \tag{6.19}$$

for $\phi_1, \phi_2 \in B_\varepsilon$ a ball of radius $\mathcal{O}(\varepsilon^{2-\frac{\sigma}{\sqrt{2}}})$ in the norm $\|\cdot\|_{2,p,2-\frac{\sigma}{\sqrt{2}},\sigma}$. A direct application of the contraction mapping principle allows us to solve the projected system

$$\partial_{tt}\phi + \Delta_{\Sigma_\varepsilon}\phi + F'(w(t))\phi = -\widetilde{S}(\mathrm{w}_\varepsilon) - \mathbf{N}(\phi) + c(y)w'(t), \quad \text{in } \Sigma_\varepsilon \times \mathbb{R}. \tag{6.20}$$

$$\int_{\mathbb{R}} \phi\,(y,t)w'(t)dt \;=\; 0, \quad y \in \Sigma_\varepsilon. \tag{6.21}$$

where

$$c(y) \;=\; \int_{\mathbb{R}} \left[\widetilde{S}(\mathrm{w}_\varepsilon) + \mathbf{N}(\phi)\right] w'(t)dt.$$

This solution ϕ, defines a Lipschitz operator $\phi = \Phi(h)$. This information is collected in the following proposition.

Proposition 6.2. *Assume $0 < \mu \le 2$, $0 < \sigma < \sqrt{2}$ and $p > 2$. For every $\varepsilon > 0$ small enough, there exists an universal constant $C > 0$, such that system (6.20)-(6.21) has a unique solution $\phi = \Phi(h)$, satisfying*

$$\|\Phi\|_{2,p,2-\frac{\sigma}{\sqrt{2}},\sigma} \le C\varepsilon^{2-\frac{\sigma}{\sqrt{2}}}$$

and

$$\|\Phi(h) - \Phi(\hat{h})\|_{2,p,2-\frac{\sigma}{\sqrt{2}},\sigma} \le C\,\varepsilon^{2-\tau}\|h - \hat{h}\|_{2,p,2+\beta}$$

for some fixed $\beta > 0$ small.

At point we remark that to complete the proof of Theorem 1, it remains to adjust the nodal set, using the parameter function h, in such a way that the coefficient $c(y)$ in the projected equation (6.20) becomes identically zero. This task is pursued in the next subsection.

6.2 Adjusting the nodal sets. First, we estimate the size of the error of the projected problem. For that we estate the following Lemma.

Lemma 6.1. *Assume $\Psi(y,t)$ is a function defined in $\Sigma_\varepsilon \times \mathbb{R}$ and for which*

$$\sup_{(y,t)\in\Sigma_\varepsilon\times\mathbb{R}} (1 + r(\varepsilon y)^\mu)e^{\sigma|t|}\|\Psi\|_{L^p(B_1(y,t))} < \infty$$

for some $\sigma, \mu > 0$ and $p > 2$. The function defined in Σ as

$$g(y) := \int_{\mathbb{R}} \Psi\left(\frac{y}{\varepsilon}, t\right) w'(t)dt$$

satisfies the estimate

$$\|g\|_{p,\mu} \le C\varepsilon^{-\frac{N}{p}} \sup_{(y,t)\in\Sigma_\varepsilon\times\mathbb{R}} (1 + r(y)^\mu)e^{\sigma|t|}\|\Psi\|_{L^p(B_1(y,t))}.$$

Proof. It is enough to notice that

$$\begin{aligned}
\int_{R-1<|y|<R+1} |g(y)|^p dV_\Sigma &= \int_{R-1<|y|<R+1} \left(\int_{\mathbb{R}} |\Psi\left(\frac{y}{\varepsilon}, t\right) w'(t)dt\right)^p dV_\Sigma \\
&\le C \int_{R-1<|y|<R+1} \int_{\mathbb{R}} \left|\Psi\left(\frac{y}{\varepsilon}, t\right)\right|^p |w'(t)|^p dt dV_\Sigma \\
&\le C\,\varepsilon^N \int_{\frac{R-1}{\varepsilon}<|\hat{y}|<\frac{R+1}{\varepsilon}} \int_{\mathbb{R}} |\Psi\,(\hat{y}, t)|^p\,|w'(t)|^p dt\, dV_{\Sigma_\varepsilon}
\end{aligned}$$

$$\leq C\varepsilon^N \sum_{j\geq N_{0,\varepsilon}}^{N_{1,\varepsilon}} \sum_{|k|\geq 0} \int_{j-1<|\hat{y}|<j+1} \int_{|t-k|<1} |\Psi(\hat{y},t)|^p |w'(t)|^p dt\, dV_{\Sigma_\varepsilon}$$

where $N_{0,\varepsilon} := \lfloor \frac{R-1}{\varepsilon} \rfloor$ and $N_{1,\varepsilon} := \lceil \frac{R+1}{\varepsilon} \rceil$. Since the set $\{\hat{y} : j-1 < |\hat{y}| < j+1\}$ can be cover with $\mathcal{O}(j^{-N})$ balls of radius 1, we conclude that

$$\int_{R-1<|y|<R+1} |g(y)|^p dV_\Sigma \leq C\varepsilon^{N-\mu p} \,\|\Psi\|^p_{p,\mu,\sigma} \sum_{j\geq N_{0,\varepsilon}}^{N_{1,\varepsilon}} \sum_{|k|\geq 0} e^{-\sigma\, p\, |k|} |j|^{-\mu p + N}$$

$$\leq C\varepsilon^{N-\mu p} \|\Psi\|^p_{p,\mu,\sigma} \sum_{j=N_{0,\varepsilon}}^{N_{1,\varepsilon}} \int_{j<|\hat{y}|<j+1} |\hat{y}|^{-\mu p+1} dV_{\Sigma_\varepsilon}$$

$$\leq C\varepsilon^{N-\mu p} \|\Psi\|^p_{p,\mu,\sigma} \int_{\frac{R-1}{\varepsilon}<|\hat{y}|<\frac{R+1}{\varepsilon}} |\hat{y}|^{1-\mu p} dV_{\Sigma_\varepsilon}$$

so that

$$\|g\|_{p,\beta} \leq C\varepsilon^{-\frac{1}{p}} \|\Psi\|_{p,\mu,\sigma}$$

for $\beta \leq \mu - \frac{N}{p}$. □

To conclude the proof of Theorem 1, we choose the function h in such a way that

$$c(y) = \int_{\mathbb{R}} \left[\widetilde{S}(\mathrm{w}_\varepsilon) + \mathbf{N}(\Phi) \right] w'(t) dt = 0, \quad y \in \Sigma_\varepsilon.$$

Let us write

$$\mathrm{Q}_1(h) := \varepsilon^{-2} \int_{\mathbb{R}} \widetilde{S}(\mathrm{w}_\varepsilon) w'(t) dt, \quad \mathrm{Q}_2(h) := \varepsilon^{-2} \int_{\mathbb{R}} \mathbf{N}(\Phi) w'(t) dt.$$

From (6.18), (6.19) and lemma 6.1 we find for $\tau = \frac{\sigma}{\sqrt{2}}$ that

$$\|\mathrm{Q}_2(h)\|_{p,4-\tau-\frac{N}{p}} \leq C\varepsilon^{1+\tau-\frac{1}{p}}$$

and

$$\|\mathrm{Q}_2(h) - \mathrm{Q}_2(\hat{h})\|_{p,4-\frac{N}{p}} \leq C\varepsilon^{1+\tau-\frac{1}{p}} \|h - \hat{h}\|_{2,p,2+\beta}.$$

Next, we analyze the term $\mathrm{Q}_1(h)$. Using the decomposition in (5.7) we set

$$c_0 = \|w'\|^2_{L^2(\mathbb{R})}, \quad a_0 = \|w'\|^{-2}_{L^2} \int_{\mathbb{R}} 6(1 - w^2(t)) e^{\sqrt{2}t} w'(t) dt$$

and evaluating at $\varepsilon^{-1} y$ we find that

$$\varepsilon^2\, \mathrm{Q}_1(h) = \int_{\mathbb{R}} \widetilde{S}(\mathrm{w}_\varepsilon) w'(t) dt \;=$$

$$-\, c_0 \varepsilon^2 \left(\Delta_\Sigma h \,+ |A_\Sigma|^2\, h \right) \,+\; 2\sqrt{2}\, a_0 e^{\sqrt{2}t}\, \chi_\varepsilon\, e^{-2\sqrt{2}\varepsilon^{-1} F_\varepsilon}\, h - c_0 \chi_\varepsilon\, E_0(r) w'(t)$$

$$-2\sqrt{2} c_0\, a_0 \chi_\varepsilon(\varepsilon r)\, e^{-2\sqrt{2}\varepsilon^{-1} F_\varepsilon} \left[e^{-2\sqrt{2}h} \,-\, 1 \,+ 2\sqrt{2}\, h \right] + \int_{\mathbb{R}} \zeta_1 R\, w'(t) dt.$$

We know from (5.14) that

$$\|E_0\|_{p,4} \leq C\, \varepsilon^4.$$

On the other hand, if h is such that $\|h\|_{2,p,2+\beta} \leq C\varepsilon^\tau$, then

$$\left| 2\sqrt{2} c_0\, a_0 \chi_\varepsilon(\varepsilon r)\, e^{-2\sqrt{2}\varepsilon^{-1} F_\varepsilon} \left[e^{-2\sqrt{2}h} \,-\, 1 \,+ 2\sqrt{2}\, h \right] \right| \leq C\varepsilon^2 (1+r)^{-2} h^2$$

so that

$$\| -2\sqrt{2} c_0\, a_0 \chi_\varepsilon\, e^{-2\sqrt{2}\varepsilon^{-1} F_\varepsilon} \left[e^{-2\sqrt{2}h} \,-\, 1 \,+ 2\sqrt{2}\, h \right] \|_{p,2+\beta} \leq C\varepsilon^{2\tau}, \quad \tau > 0.$$

and the Lipschitz dependence of this term respect to h and in the topology described above, holds true with Lipschitz constant $\mathcal{O}(\varepsilon^{2+\tau})$.

Finally, from (6.12) it is straight forward to check that

$$\left\| \int_{\mathbb{R}} \zeta_1 R w'(t) dt \right\|_{p,2+\beta} \leq C\varepsilon^{2+\tau_1}, \quad \tau_1 > \tau$$

being as well a contraction, with Lipschitz constant $\mathcal{O}(\varepsilon^{2+\tau_1})$.

Hence applying Proposition (4.1) and a fixed point argument for h in the ball

$$B := \{h \in W^{2,p}_{loc}(\Sigma) : \|h\|_{2,p,2+\beta} \leq C\varepsilon^{\tau}\}, \quad p > N$$

we find h in such a way that $c(y) = 0$, and this finishes the proof of Theorem 1. The next section is devoted to provide the proofs of the propositions and lemmas mentioned here.

7. Gluing reduction and solution to the projected problem.

In this section, we prove propositions 6.1 and 6.2. The notations we use in this section have been set up in sections 4 and 5.

7.1 Solving the Gluing System. Given a fixed function ϕ such that $\|\phi\|_{2,p,\mu,\sigma} \leq 1$, we solve problem (6.9). To begin with, we observe that setting

$$Q_\varepsilon(x) \;=\; 2 - (1-\zeta_2)\,[F'(\mathrm{w}_\varepsilon) + 2]\,.$$

there exist constants $a < b$, independent of ε, such that

$$0 < a \leq Q_\varepsilon(x) \leq b, \quad \text{ for every } x \in \mathbb{R}^{N+1}.$$

Using this remark, we study the problem

$$\Delta\psi - Q_\varepsilon(x)\psi = g(x), \quad x \in \mathbb{R}^{N+1} \tag{7.1}$$

for a given $g = g(x)$ such that

$$\|g\|_{p,\hat{\mu},\sim} := \sup_{x \in \mathbb{R}^{N+1}} (1 + R^{\hat{\mu}}(\varepsilon x))\|g\|_{L^p(B_1(x))}.$$

Solvability theory for equation (7.1) is collected in the following lemma whose proof follows the same lines as in lemma 7.1 in [13] and [16].

Lemma 7.1. *Assume $p > 2$ and $\hat{\mu} \geq 0$. There exists a constant $C > 0$ and $\varepsilon_0 > 0$ small enough such that for $0 < \varepsilon < \varepsilon_0$ and any given $g = g(x)$ with $\|g\|_{p,\hat{\mu},\sim} < \infty$, equation (7.1) has a unique solution $\psi = \psi(g)$, satisfying the a-priori estimate*

$$\|\psi\|_X \leq C\|g\|_{p,\hat{\mu},\sim}$$

where

$$\|\psi\|_X := \|D^2\psi\|_{p,\hat{\mu},\sim} + \|(1 + r(\alpha x)^{\hat{\mu}}(x))\, D\psi\|_{L^\infty(\mathbb{R}^{N+1})} + \|(1 + r^{\hat{\mu}}(\alpha x))\,\psi\|_{L^\infty(\mathbb{R}^{N+1})}.$$

Now we prove Proposition 6.1. Denote by X, the space of functions $\psi \in W^{2,p}_{loc}(\mathbb{R}^{N+1})$ such that $\|\psi\|_X < \infty$ and let us denote by $\Gamma(g) = \psi$ the solution to the equation (7.1) from the previous lemma. We see that the linear map Γ is continuous i.e

$$\|\Gamma(g)\|_X \leq C\|g\|_{p,\hat{\mu},\sim}$$

with $0 < \hat{\mu} < \min(2\mu\,,\, \mu + \varrho\sqrt{2}\,,\, 2 + \varrho\sqrt{2})$. Using this we can recast (6.9) as a fixed point problem, in the following manner

$$\psi = -\Gamma\left((1-\zeta_2)\,S(\mathrm{w}_\varepsilon) \,+\, \nabla\zeta_2 \cdot \nabla\phi \,+\, \phi\Delta\zeta_2 \,+\, (1-\zeta_2)\,N\,[\zeta_3\phi \,+\, \psi]\right) \tag{7.2}$$

Under conditions (3.23) and $\|\phi\|_{2,p,\mu,\sigma} \leq 1$, we estimate the size of the right-hand side in (7.2). Also recall from (6.13)

$$\|\widetilde{S}(\mathrm{w}_\varepsilon)\|_{p,2-\frac{\sigma}{\sqrt{2}},\sigma} \leq C\varepsilon^{2-\frac{\sigma}{\sqrt{2}}}$$

for any $\sigma \in (0,\sqrt{2})$. Since the support of ζ_2 is contained in a region of the form

$$\varrho_\varepsilon - 1 \leq |t+h| \leq \varrho_\varepsilon + 1$$

where

$$\varrho_\varepsilon(x) = \frac{\delta_0}{\varepsilon}\left(1-\eta_\varepsilon(\varepsilon r)\right) + \eta_\varepsilon(\varepsilon r)\varepsilon^1 F_\varepsilon(\varepsilon r) - 1, \quad y = Y_\varepsilon(r,\Theta) \in \Sigma_\varepsilon$$

we estimate directly for $0 < \hat{\mu} \leq 2$, to get that

$$\|(1-\zeta_2)\, S(\mathrm{w}_\varepsilon)\|_{p,\hat{\mu},\sim} \;\leq\; C\varepsilon^2$$

for some $\varepsilon > 0$ sufficiently small.

As for the second term in the right-hand side of (7.2), the following holds true

$$\begin{aligned} |2\nabla\zeta_2 \cdot \nabla\phi + \phi\Delta\zeta_2| &\leq C(1-\zeta_2)(1+r^\mu(\varepsilon y))^{-1}e^{-\sigma|t|}\|\phi\|_{2,p,\mu,\sigma} \\ &\leq \quad C\varepsilon^{\frac{\sigma}{\sqrt{2}}}(1+r^{\mu+\frac{\sigma}{\sqrt{2}}}(\varepsilon r))^{-1}\|\phi\|_{2,p,\mu,\sigma}. \end{aligned}$$

This implies that

$$\|2\nabla\zeta_2 \cdot \nabla\phi + \phi\Delta\zeta_2\|_{p,\mu+\frac{\sigma}{\sqrt{2}},\sim} \leq C\varepsilon^{\frac{\sigma}{\sqrt{2}}}\|\phi\|_{2,p,\mu,\sigma}.$$

Finally we must check the lipschitz character of $(1-\zeta_2)N[\zeta_2\phi+\psi]$. Take $\psi_1,\psi_2 \in X$. Then

$$\begin{gathered} (1-\zeta_2)\,|N\,[\zeta_2\phi+\psi_1] - N\,[\zeta_2\phi+\psi_2]| \leq \\ \leq (1-\zeta_2)\,|F(\mathrm{w}+\zeta_1\phi+\psi_1) - F(\mathrm{w}+\zeta_1\phi+\psi_2) - F'(\mathrm{w})(\psi_1-\psi_2)| \\ \leq \quad C\,(1-\zeta_2)\sup_{s\in[0,1]}|\zeta_1\phi + s\psi_1 + (1-s)\psi_2|\,|\psi_1-\psi_2|. \end{gathered}$$

So, we see that

$$\begin{gathered} \|(1-\zeta_2)\,N\,[\zeta_2\phi+\psi_1] - (1-\zeta_2)\,N\,[\zeta_2\phi+\psi_2]\|_{p,2\hat{\mu},\sim} \\ \leq C\varepsilon^{\frac{\sigma}{\sqrt{2}}}\left(\|\phi\|_{\infty,\mu,\sigma} + \|\psi_1\|_X + \|\psi_2\|_X\right)\|\psi_1-\psi_2\|_{\infty,\hat{\mu},\sim} \\ \leq C\varepsilon^{\frac{\sigma}{\sqrt{2}}}\|\psi_1-\psi_2\|_{\infty,\hat{\mu},\sim}. \end{gathered}$$

In particular, we take advantage of the fact that $N(\varphi) \sim \varphi^2$, to find that there exists $\alpha > 0$ small such that

$$\|(1-\zeta_2)\,N\,(\zeta_2\phi)\|_{p,2\mu,\sim} \leq C\varepsilon^{\frac{\sigma}{\sqrt{2}}-\alpha}\|\phi\|^2_{2,p,\mu,\sigma}\,.$$

Consider $\widetilde{\Gamma} : X \to X$, $\widetilde{\Gamma} = \widetilde{\Gamma}(\psi)$ the operator given by the right-hand side of (7.2). From the previous remarks we have that $\widetilde{\Gamma}$ is a contraction provided $\varepsilon > 0$ is small enough and so we have found $\psi = \widetilde{\Gamma}(\psi)$ the solution to (6.9) with

$$\|\psi\|_X \leq C\left(\varepsilon^2 + \varepsilon^{\frac{\sigma}{\sqrt{2}}}\|\phi\|_{2,p,\mu,\sigma}\right).$$

We can check directly that $\Psi(\phi) = \psi$ is Lipschitz in ϕ, i.e

$$\begin{gathered} \|\Psi(\phi_1) - \Psi(\phi_2)\|_X \leq \\ C\,\|(1-\zeta_2)\,[N\,(\zeta_2\phi_1 + \Psi(\Phi_1)) - N\,(\zeta_2\phi_2 + \Psi(\phi_2))]\|_{p,2\mu,\sim} \\ \leq C\varepsilon^{\frac{\sigma}{\sqrt{2}}}\left(\|\Psi(\phi_1)-\Psi(\phi_2)\|_X + \|\phi_1-\phi_2\|_{2,p,\mu,\sigma}\right) \end{gathered}$$

so that for ε small, we conclude

$$\|\Psi(\phi_1)-\Psi(\phi_2)\|_X \leq C\varepsilon^{\tau}\|\phi_1-\phi_2\|_{2,p,\mu,\sigma}.$$

7.2 Solving the Projected equation (6.20)-(6.21)**.** Now we solve the the projected problem

$$\partial_{tt}\phi+\Delta_{\Sigma_\varepsilon}\phi+F'(w(t))\phi = -\widetilde{S}(\mathrm{w})-\mathbf{N}(\phi)\,+\,c(y)w'(t), \quad \text{in } \Sigma_\varepsilon\times\mathbb{R}.$$

$$\int_{\mathbb{R}}\phi(y,t)w'(t)dt\,=\,0.$$

To do so, we need to study solvability for the linear equation

$$\partial_{tt}\phi+\Delta_{\Sigma_\varepsilon}\phi+F'(w(t))\phi = g(y,t)+c(y)\,w'(t), \quad \text{in } \Sigma_\varepsilon\times\mathbb{R} \tag{7.3}$$

$$\int_{\mathbb{R}}\phi(y,t)w'(t)dt = 0. \tag{7.4}$$

Solvability of (7.3)-(7.4) is based upon the fact that the heteroclinic solution $w(t)$ is nondegenerate in the sense, that the following property holds true.

Lemma 7.2. *Assume that $\phi\in L^\infty(\mathbb{R}^{N+1})$ and assume $\phi=\phi(x',t)$ satisfies*

$$L(\phi):=\partial_{tt}\phi+\Delta_{\mathbb{R}^N}\phi+F'(w(t))\phi=0, \qquad \text{in } \mathbb{R}^N\times\mathbb{R}. \tag{7.5}$$

Then $\phi(x',t)\,=\,C\,w'(t)$, for some constant $C\in\mathbb{R}$.

For the detailed proof of this lemma we refer the reader to [13], [16] and references therein.

The linear theory we need to solve system (6.21), is collected in the following proposition, whose proof is again contained in essence in proposition 4.1 in [13] and [16].

Proposition 7.1. *Assume $p>2$, $0<\sigma<\sqrt{2}$ and $\mu\geq 0$. There exist $C>0$, an universal constant, and $\varepsilon_0>0$ small such that, for every $\varepsilon\in(0,\varepsilon_0)$ and any given g with $\|g\|_{p,\mu,\sigma}<\infty$, problem* (7.3)-(7.4) *has a unique solution (ϕ,c) with $\|\phi\|_{p,\mu,\sigma}<\infty$, satisfying the apriori estimate*

$$\|D^2\phi\|_{p,\mu,\sigma}+\|D\phi\|_{\infty,\mu,\sigma}+\|\phi\|_{\infty,\mu,\sigma}\leq C\|g\|_{p,\mu,\sigma}.$$

Using Proposition 7.1, we are ready to solve system (6.20)-(6.21). First, recall that as stated in (6.13)

$$\|\widetilde{S}(\mathrm{w}_\varepsilon)\|_{p,2-\frac{\sigma}{\sqrt{2}},\sigma}\leq C\varepsilon^{2-\frac{\sigma}{\sqrt{2}}}. \tag{7.6}$$

From proposition 6.1 we have a nonlocal operator $\psi=\Psi(\phi)$. We want to solve the following problem

Recall that

$$\mathbf{N}(\phi):=\,B(\phi)+[F'(\zeta_2\mathrm{w}_\varepsilon)-F'(w(t))]\,\phi\,+$$

$$+\,\zeta_2[F'(\mathrm{w}_\varepsilon)+2]\Psi(\phi)+\zeta_2 N(\phi+\Psi(\phi)).$$

Let us denote

$$N_1(\phi):=\,B(\phi)+[F'(\zeta_2\mathrm{w}_\varepsilon)-F'(w(t))]\,\phi$$

$$N_2(\phi):=\,\zeta_2\,[F'(\mathrm{w}_\varepsilon)+2]\,\Psi(\phi)$$

$$N_3(\phi):=\zeta_2 N(\phi+\Psi(\phi)).$$

We need to investigate the Lipschitz character of N_i, $i=1,2,3$. We begin with N_3. Observe that

$$|N_3(\phi_1)-N_3(\phi_2)|=\zeta_2|N(\phi_1+\Psi(\phi_1))-N(\phi_2+\Psi(\phi_2))|$$

$$\leq C\zeta_2 \sup_{\tau\in[0,1]} |\tau(\phi_1+\Psi(\Phi_1))+(1-\tau)(\phi_2+\Psi(\phi_2))|\cdot|\phi_1-\phi_2+\Psi(\phi_1)-\Psi(\phi_2)|$$

$$\leq C\left[|\Psi(\Phi_2)|+|\phi_1-\phi_2|+|\Psi(\phi_1)-\Psi(\phi_2)|+|\phi_2|\right]\cdot\left[|\phi_1-\phi_2|+|\Psi(\phi_1)-\Psi(\phi_2)|\right].$$

This implies that

$$\|N_3(\phi_1)-N_3(\phi_2)\|_{p,2\mu,\sigma}\leq$$
$$\leq C\left[\varepsilon^2+\|\phi_1\|_{\infty,\mu,\sigma}+\|\phi_2\|_{\infty,\mu,\sigma}\right]\cdot\|\phi_1-\phi_2\|_{\infty,\mu,\sigma}.$$

Now we check on $N_1(\phi)$. Clearly, we just have to pay attention to $B(\phi)$. But notice that $B(\phi)$ is linear on ϕ and

$$B(\phi)=-\varepsilon^2\left\{\Delta_\Sigma h+|A_\Sigma|^2(t+h)\right\}\partial_t\phi$$
$$-2\varepsilon\nabla_\Sigma h(\varepsilon y)\partial_t\nabla_{\Sigma_\varepsilon}\phi+\varepsilon^2[\nabla h(\varepsilon y)]^2\partial_{tt}\phi+D_{\varepsilon,h}(\phi),$$

where the differential operator $D_{\varepsilon,h}$ is given in (3.25). From assumptions (3.23) made on the function h, we have that

$$\|N_1(\phi_1)-N_1(\phi_2)\|_{p,1+\mu,\sigma}\leq C\varepsilon\|\phi_1-\phi_2\|_{2,p,\mu,\sigma}.$$

Then, assuming that $\|\phi\|_{2,p,\mu,\sigma}\leq A\varepsilon^{2-\frac{\sigma}{\sqrt{2}}}$, we have that

$$\|\mathbf{N}(\phi)\|_{p,1+\mu,\sigma}\leq C\varepsilon^{3-\frac{\sigma}{\sqrt{2}}}$$

Setting $T(g)=\phi$ the linear operator given by the Lemma 7.1, we recast problem (6.20) as the fixed point problem

$$\phi=T(-\widetilde{S}(\mathrm{w}_\varepsilon)-\mathbf{N}(\phi))=:\mathcal{T}(\phi)$$

in the ball

$$B_\varepsilon:=\left\{\phi\in W^{2,p}_{loc}(\Sigma_\varepsilon\times\mathbb{R}):\|\phi\|_{2,p,\mu,\sigma}\leq A\,\varepsilon^{2-\frac{\sigma}{\sqrt{2}}}\right\}$$

where $0<\mu\leq 2-\frac{\sigma}{\sqrt{2}}$.

Observe that

$$\|\mathcal{T}(\phi_1)-\mathcal{T}(\phi_2)\|_{2,p,\mu+1,\sigma}\leq C\,\|\mathbf{N}(\phi_1)-\mathbf{N}(\phi_2)\|_{p,\mu+1,\sigma}\leq C\,\varepsilon\|\phi_1-\phi_2\|_{2,p,\mu,\sigma},\quad \Phi_1,\phi_2\in B_\varepsilon.$$

On the other hand, because C and K_1 are universal constants and taking A large enough independent of $\varepsilon>0$, we have that

$$\|\mathcal{T}(\phi)\|_{2,p,\mu,\sigma}\leq C\,\left(\|\widetilde{S}(\mathrm{w}_\varepsilon)\|_{p,2-\frac{\sigma}{\sqrt{2}},\sigma}\,+\,\|\mathbf{N}(\phi)\|_{p,4,\sigma}\right)\leq\,A\,\varepsilon^{2-\frac{\sigma}{\sqrt{2}}},\quad \phi\in B_\varepsilon.$$

Hence, the mapping $\mathcal{T}$ is a contraction from the ball B_ε onto itself. From the contraction mapping principle we get a unique solution

$$\phi=\Phi(h)$$

as required. As for the Lipschitz character of $\Phi(h)$ it comes from a lengthy by direct computation from the fact that

$$\|\Phi(h)-\Phi(\widetilde{h})\|_{2,p,2-\frac{\sigma}{\sqrt{2}},\sigma}$$

$$\leq C\|\widetilde{S}(\mathrm{w}_\varepsilon,h)-\widetilde{S}(\mathrm{w}_\varepsilon,\widetilde{h})\|_{p,2-\frac{\sigma}{\sqrt{2}},\sigma}+$$

$$+\|N(\,\Phi(h)\,)-N(\,\Phi(\widetilde{h}\,)\|_{p,3-\frac{\sigma}{\sqrt{2}},\sigma}.$$

We left to the reader to check on the details of the proof of the following estimate

$$\|\Phi(h)-\Phi(\widetilde{h})\|_{2,p,2-\frac{\sigma}{\sqrt{2}}\,\sigma}\sqrt{2}\leq C\varepsilon^{2-\frac{\sigma}{\sqrt{2}}}\|h-\widetilde{h}\|_{p,\beta}$$

for h and $\widetilde{h}$ satisfying (3.23). This completes the proof of proposition 6.2 and consequently the proof of Theorem 1.

8. The proof of Theorem 2

8.1. **Remarks on the profile of the solutions found in Theorem 1:** In order to proof our second result, we need further information about the asymptotic profile of the solutions predicted in Theorem 1, as the parameter $\varepsilon > 0$ approaches zero.

Let us focus for the moment in the set $\Sigma_{\varepsilon,+}$. Taking a closer look to the error (6.12) of the initial approximation $\bar{u}$, described in section 5, we observe that the

$$
\begin{aligned}
S(\bar{u}) \,=\, & -\varepsilon^2 \left\{\Delta_\Sigma h \,+ |A_\Sigma|^2\, h\right\} w'(t) \,-\, 6\left(1-w^2(t)\right) e^{\sqrt{2}t}\, \chi_\varepsilon\, 2\sqrt{2}\, e^{-2\sqrt{2}\varepsilon^{-1}F_\varepsilon}\, h \\
& -\chi_\varepsilon\, E_0(\varepsilon r) w'(t) \,+\, g_0(t) e^{-2\sqrt{2}\varepsilon^{-1}F_\varepsilon} \,-\, \varepsilon^2 |A_\Sigma|^2 t\, w'(t) \,+\, \varepsilon^2\, |\nabla_\Sigma h|^2\, w''(t) \\
& -6(1-w^2(t)) e^{\sqrt{2}t} \chi_\varepsilon(\varepsilon r)\, e^{-2\sqrt{2}\varepsilon^{-1}F_\varepsilon} \left[e^{-2\sqrt{2}h} \,-\, 1 \,+ 2\sqrt{2}\, h\right] + R.
\end{aligned}
$$

and where

$$
|D_p R(\varepsilon r, t, p, q)| \,+\, |D_q R(\varepsilon r, t, p, q)| \,+\, |R(\varepsilon r', t, p, q)| \leq C\varepsilon^{2-\tau}(1+|\varepsilon r|)^{-2+\tau} e^{-\sigma|t|}
$$

for some $0 < \sigma < \sqrt{2}$ and some $\tau = \frac{\sigma}{\sqrt{2}}$.

Let us consider $\psi_0(t)$ to be the bounded solution to the equation

$$
\partial_{tt}\psi_0(t) + F'(w(t))\psi_0(t) = g_0(t), \quad t \in \mathbb{R} \tag{8.1}
$$

given explicitly by the variations of parameters formula

$$
\psi_0(t) = w'(t) \int_0^t w'(s)^{-2} \int_s^\infty w'(\xi) g_0(\xi)\, d\xi\, ds. \tag{8.2}
$$

From (8.2), we obtain the estimate

$$
\left\| \left(1 \,+\, e^{2\sqrt{2}\, t} \chi_{\{t>0\}}\right) \partial_t^{(j)} \psi_0 \right\|_{L^\infty(\mathbb{R})} \leq C_j, \quad j \in \mathbb{N}.
$$

Let us also consider the function $\psi_1(t)$ solving

$$
\partial_{tt}\psi_1(t) + F'(w(t))\psi_1(t) = t w'(t), \quad t \in \mathbb{R}. \tag{8.3}
$$

Proceeding as before, we see that

$$
\psi_1(t) = -w(t) \int_0^t w'(s)^{-2} \int_s^\infty \xi w'(\xi)^2\, d\xi ds
$$

and $\psi_1(t) = -\frac{1}{2} t w'(t)$, from where the following estimate follows at once

$$
\| e^{\sigma|t|} \partial_t^{(j)} \psi_1 \|_{L^\infty(\mathbb{R})} \leq C_j, \quad j \in \mathbb{N}, \quad 0 < \sigma < \sqrt{2}.
$$

So, we take the function

$$
\bar{\mathrm{w}}_\varepsilon(x) \,=\, \mathrm{w}_\varepsilon \,+\, \phi_0 \tag{8.4}
$$

as a second approximation, in a way that in the region $\mathcal{N}_{\varepsilon,h}$ and in the coordinates $X_{\varepsilon,h}$

$$
\phi_0(y,t) = -e^{-2\sqrt{2}\varepsilon^{-1}F_\varepsilon}\, \psi_0(t) \,+\, \varepsilon^2 |A_\Sigma(\varepsilon y)|^2\, \psi_1(t)
$$

for the upper hemisphere, while in the lower part, the function ϕ_0 is described as

$$
\phi_0(y,t) = -e^{-2\sqrt{2}\varepsilon^{-1}F_\varepsilon}\, \psi_0(-t) \,+\, \varepsilon^2 |A_\Sigma(\varepsilon y)|^2\, \psi_1(t).
$$

Proceeding as we did to verify (5.16) we verify that the error created by $\bar{\mathrm{w}}_\varepsilon$ has the size

$$
\|S(\bar{\mathrm{w}}_\varepsilon)\|_{p,2,\sigma} \leq C\varepsilon^{2+\tau}
$$

and we can run the reduction procedure in the exact fashion as we did in section 6 to find that our solutions u_ε having in the region $\mathcal{N}_{\varepsilon,h}$ the asymptotic profile

$$
u_\varepsilon \,=\, \bar{u} + \phi_0 + \phi_\varepsilon
$$

for a function ϕ_ε such that

$$\|\phi_\varepsilon\|_{2,p,2,\sigma} \le C\varepsilon^{2+\tau}, \quad \text{for } 0<\tau<\sqrt{2}.$$

8.2. **Energy and Morse Index estimates:** Estimate the Morse Index of the solutions described in 1 are deeply connected with the number of negative eigenvalues for the linear Hardy equation

$$\Delta_{\mathbb{R}^N}\mathrm{k} + (2(N-2)+\lambda)r^{-2}\mathrm{k} = 0, \quad \text{in } \{R_\varepsilon < r < \infty\}, \quad \mathrm{k}\in L^\infty(\{r>R_\varepsilon\}). \tag{8.5}$$

Our analysis is based upon the fact that test functions of the form

$$v(x', x_{N+1}) \approx k(y)\left[w'(t) - w(t-2\varepsilon^{-1}F_\varepsilon - 2h)\right], \quad \text{for } r(\varepsilon y) > R_\varepsilon.$$

with $k(y) = \mathrm{k}(\varepsilon y)$, for $y\in\Sigma_\varepsilon$ and k as in (8.5), are negative direction for the quadratic form

$$Q(v,v) := \iint_{W_{R_\varepsilon}} |\nabla v|^2 - F'(u_\varepsilon)v^2 dx$$

Due to the symmetries considered in our discussion, we focus our analysis in the upper part of the surface Σ_ε. The exact same analysis and computations hold true in the lower part of the dilated surface.

Let k be a smooth, axially symmetric and compactly supported function defined in Σ_ε whose support is contained in the set $\{r(\varepsilon y) > R_\varepsilon\}$. In the set $\Sigma_{\varepsilon,h,+}$ where $z = t + h(\varepsilon y)$, we consider a test function of the form

$$v(x', x_{N+1}) = k(y)\left[w'(t) - w(t-2\varepsilon^{-1}F_\varepsilon - 2h)\right], \quad \text{for } r(\varepsilon y) > R_\varepsilon. \tag{8.6}$$

To compute the euclidean Laplacian for v in $\mathcal{N}_{\varepsilon,h}$ we notice first from (8.4) that in the upper part of the dilated surface Σ_ε and in the coordinates $X_{\varepsilon,h}$

$$\begin{aligned} F'(u_\varepsilon)w'(t) = F'(w)w' + &\\ F''(w)w'\left[(-2\,e^{\sqrt{2}t} + \psi_0(t))\,e^{-2\sqrt{2}(\varepsilon^{-1}F_\varepsilon + h)} + \varepsilon^2|A_\Sigma|^2\psi_1(t)\right]&\\ + \mathcal{O}\left(\varepsilon^{2+\tau}(1+r(\varepsilon y)^2)^{-1}e^{-\sigma|t|}\right).& \end{aligned} \tag{8.7}$$

As for the interaction term we compute

$$F'(u_\varepsilon)w'(t-2\varepsilon^{-1}F_\varepsilon - 2h) = F'(w(t-2\varepsilon^{-1}F_\varepsilon-2h))w'(t-2\varepsilon^{-1}F_\varepsilon-2h)$$

$$+\sqrt{2}(F'(w)-F'(1))e^{\sqrt{2}t}e^{-2\sqrt{2}(\varepsilon^{-1}F_\varepsilon+h)} + \mathcal{O}\left(\varepsilon^{2+\tau_1}e^{-\sigma}|t|(1+r(\varepsilon y))^{-2+\beta}\right)$$

for some $\beta > 0$.

Hence, from expression (3.24) and the above comments

$$\Delta_{X_{\varepsilon,h}}v + F'(u_\varepsilon)v =$$

$$\underbrace{\Delta_{\Sigma_\varepsilon}k\,w' - \varepsilon^2|A_\Sigma|^2k\,tw'' + \varepsilon^2|\nabla h|^2k\,w''' + \varepsilon a_1\,h\,\nabla_{\Sigma_\varepsilon}k\,w'}_{Q_1}$$

$$+\underbrace{F''(w(t))\left[(-2e^{\sqrt{2}t} - \psi_0(t))e^{-2\sqrt{2}(\varepsilon^{-1}F_\varepsilon+h)} + \varepsilon^2|A_\Sigma|^2\psi_1(t)\right]k\,w'}_{Q_2}$$

$$\underbrace{\sqrt{2}(F'(w)-F'(1))e^{\sqrt{2}t}e^{-2\sqrt{2}(\varepsilon^{-1}F_\varepsilon+h)}k}_{Q_3}$$

$$\underbrace{-w''\ \left[\varepsilon^2 \mathcal{J}_\Sigma(h)k \,+\, 2\varepsilon\, \nabla_\Sigma h\, \nabla_{\Sigma_\varepsilon} k \,+\, \varepsilon^2\, a_1\, h(\nabla_\Sigma h \nabla_\varepsilon k + \Delta_\Sigma h\, k\,)\right]}_{Q_4}$$

$$\underbrace{\varepsilon\, t w' \left[a_1 \Delta_{\Sigma_\varepsilon} k + \varepsilon b_1 \nabla_{\Sigma_\varepsilon} k\right]}_{Q_5} + \underbrace{\mathcal{O}\left(\varepsilon^{2+\tau_1}(1+r(\varepsilon y)^{2+\tau_1})^{-1} e^{-\sigma|t|}\right)}_{Q_6} \tag{8.8}$$

for any $\tau_1 < \tau$.

Since $F''(w) = -6w$, taking derivatives in equations (8.1) and (8.3) and integrating against $w'(t)$, we easily check that

$$\int_{\mathbb{R}} F''(w)(w')^2 \psi_2(t) dt = -\int_{\mathbb{R}} (w'')^2 dt$$

$$\int_{\mathbb{R}} F''(w)(w')^2 \psi_1(t) dt = -\int_{\mathbb{R}} t w' w'' dt = \frac{1}{2}\int_{\mathbb{R}} (w')^2 dt$$

$$\int_{\mathbb{R}} F''(w)(w')^2 \left(-2e^{\sqrt{2}t} + \psi_0(t)\right) dt = \sqrt{2}\int_{\mathbb{R}} 6(1-w^2) e^{\sqrt{2}t} w' dt = \sqrt{2}\, a_0 \int_{\mathbb{R}} (w'(t))^2 dt.$$

Also, $F'(w) - F'(1) = 6(1-w^2)$,and we find that

$$\sqrt{2}\int_{\mathbb{R}} 6(1-w^2(t)) e^{\sqrt{2}t} w'(t) dt = \sqrt{2} a_0 \int_{\mathbb{R}} (w'(t))^2 dt.$$

Observe also that

$$\int_{|t|<\rho_\varepsilon} Q_i w'(t) dt = \int_{\mathbb{R}} Q_i w'(t) dt \,+\, \mathcal{O}\left(\varepsilon^{2+\tau_1}(1+r(\varepsilon y)^{2+\tau_1})^{-1}\right)$$

where

$$\rho_\varepsilon(y) := \frac{\delta_0}{\varepsilon}(1-\eta_\varepsilon) - \frac{\eta_\varepsilon}{2\sqrt{2}}\left(\log\left(\frac{2\sqrt{2}\, a_0}{\varepsilon^2}\right) + 2\log(\varepsilon r)\right).$$

We observe also that

$$\int_{\mathbb{R}} \sum_{i=1}^{5} Q_i\, w'(t) dt \;=\; \left(\Delta_{\Sigma_\varepsilon} k + \varepsilon^2 |A_\Sigma|^2 k + 2\sqrt{2} a_0 e^{-2\sqrt{2}(\varepsilon^{-1}F_\varepsilon + h)} k + \varepsilon a_1 h \partial_{ij} k\right) \int_{\mathbb{R}} w'(t)^2 dt$$

and

$$\int_{\mathbb{R}} Q_6\, w'(t) dt \;=\; \mathcal{O}(\varepsilon^2 r(\varepsilon y)^{-2}) \partial_{ij} k + \mathcal{O}\left(\varepsilon^3 r(\varepsilon y)^{-3}\right) \nabla_i k$$

Combining these computations , we have that

$$\int_{|t|\le\rho_\varepsilon} (\Delta v + F'(u_\varepsilon) v)\, w'(t) dt \;=\; \left(\Delta_{\Sigma_\varepsilon} k \,+\, \varepsilon^2 |A_\Sigma|^2 k \,+\, \varepsilon a_1 h \partial_{ij} k\right) \int_{\mathbb{R}} w'(t)^2 dt$$

$$+2\sqrt{2} a_0\, e^{-2\sqrt{2}(\varepsilon^{-1}F_\varepsilon + h)} k \int_{\mathbb{R}} w'(t)^2 dt$$

$$+\,\mathcal{O}(\varepsilon^2 r(\varepsilon y)^{-2}) D^2_{\Sigma_\varepsilon} k \,+\, \mathcal{O}\left(\varepsilon^3 r(\varepsilon y)^{-3}\right) \nabla_{\Sigma_\varepsilon} k \,+\, \mathcal{O}(\varepsilon^{2+\tau_1} r(\alpha y)^{-2-\beta}) k.$$

Regarding computations in the lower part of Σ_ε we find that

$$F'(u_\varepsilon) w' = F'(w) w' +$$

$$+\, F''(w) w' \left[\left(2\, e^{-\sqrt{2}t} - \psi_0(-t)\right) e^{-2\sqrt{2}(\varepsilon^{-1}F_\varepsilon + h)} + \varepsilon^2 |A_\Sigma|^2 \psi_2(t)\right]$$

$$+\; \mathcal{O}\left(\varepsilon^{2+\tau_1}(1+r(\varepsilon y)^2)^{-1} e^{-\sigma|t|}\right) \tag{8.9}$$

and the interaction term this time takes the form

$$F'(u_\varepsilon)w'(t-2\varepsilon^{-1}F_\varepsilon-2h)=F'(w(t-2\varepsilon^{-1}F_\varepsilon-2h))w'(t-2\varepsilon^{-1}F_\varepsilon-2h)$$

$$-\sqrt{2}(F'(w)-F'(1))e^{-\sqrt{2}t}e^{-2\sqrt{2}(\varepsilon^{-1}F_\varepsilon+h)}+\mathcal{O}\left(\varepsilon^{2+\tau_1}e^{-\sigma}|t|(1+|\varepsilon y|)^{-2-\beta}\right).$$

Similar computations as the ones above, in the region

$$W_{R_\varepsilon}=\{x\in\mathcal{N}_\varepsilon : r(\varepsilon x)>R_\varepsilon\}$$

lead to the expression

$$\begin{aligned} Q(v,v) &= \iint_{W_{R_\varepsilon}}|\nabla v|^2-F'(u_\varepsilon)v^2dx\\ &= 2\int_{\mathbb{R}}w'(t)^2dt\int_{\Sigma_\varepsilon}|\nabla_{\Sigma_\varepsilon}k|^2-\varepsilon^2|A_\Sigma|^2k^2+2\sqrt{2}a_0e^{-2\sqrt{2}(\varepsilon^{-1}F_\varepsilon+h)}k^2dV_{\Sigma_\varepsilon}\\ &\quad+\mathcal{O}\left(\varepsilon\int_{\Sigma_\varepsilon}|\nabla_\Sigma k|^2+\varepsilon^2(1+r(\varepsilon y))^{-2-\beta}k^2dV_{\Sigma_\varepsilon}\right). \end{aligned}$$

Finally, since

$$2\sqrt{2}a_0e^{-2\sqrt{2}(\varepsilon^{-1}F_\varepsilon+h)} = \frac{2(N-2)\,\varepsilon^2}{r^2}+\varepsilon^2\mathcal{O}(r^{-2}(w_\varepsilon+h))$$

we find that

$$2\sqrt{2}a_0e^{-\sqrt{2}(\varepsilon^{-1}F_\varepsilon+h)}=\varepsilon^2\left(\frac{2(N-2)}{r^2}(1+p(r))\right)$$

where $p(r)=\mathcal{O}(r^{-\beta})$ is smooth and

$$\nabla_\Sigma=\nabla_{\mathbb{R}^N}++\varepsilon\mathcal{O}(r^{-1})\partial_r.$$

$$c\,\varepsilon^2(1+r)^{-4}\le|A_\Sigma|^2\le C\,\varepsilon^2(1+r)^{-4},\quad r>R_\varepsilon.$$

Consequently, we obtain

$$Q(v,v)=\gamma_0\int_{r(\varepsilon y)>R_\varepsilon}|\nabla_{\mathbb{R}^N}k|^2-\varepsilon^2\frac{2(N-2)}{r^2}k^2d\,r\;+\;\varepsilon^{\tau_1}\mathcal{O}\left(\int_{r(\varepsilon y)>R_\varepsilon}|\nabla_{\mathbb{R}^N}k|^2+\varepsilon^2(1+\varepsilon r)^{-2-\beta})k^2\right).\tag{8.10}$$

Let $\mathrm{k}(y)$ be an axially symmetric solution of the equation

$$\Delta_{\mathbb{R}^N}\mathrm{k}\;+\;(2(N-2)+\lambda)r^{-2}\mathrm{k}=0,\quad\text{in }\{R_\varepsilon<r<\infty\}$$

with

$$\lambda=\lambda_m=\frac{1}{2}\sqrt{4m^2-(N-10)(N-2)}$$

so that we may write explicitely

$$\mathrm{k}(r)=\cos(\lambda_m\log(r))r^{-\frac{N-2}{2}}.$$

Letting $k(y)$ be axially symmetric and defined in Σ_ε in a way that in coordinates $X_{\varepsilon,h}(r,\theta,t)$ it coincides with $\mathrm{k}(\varepsilon r)$, but cut off at infinity.

With $v(x',x_{N+1})$ as in (8.6) and from (8.10), it follows that

$$\begin{aligned} Q(v,v) &= \iint_{W_R}|\nabla v|^2-F'(u_\varepsilon)v^2dx\\ &= -2\gamma_0\,\varepsilon^{-N+2}\left[m^2\int_{r(y)>R_\varepsilon}r^{-2}\mathrm{k}^2dr+\mathcal{O}\left(\varepsilon^{\tau_1}\int_{r(y)>R_\varepsilon}|\nabla_{\mathbb{R}^N}\mathrm{k}|^2+(1+r)^{-2}\mathrm{k}^2\right)\right] \end{aligned}$$

and we conclude that $Q(v,v)<0$ for every $\varepsilon>0$ small.

Since $3\le N\le 9$ and since $m\in\mathbb{N}$ is arbitrary, for every $m\in\mathbb{N}$, we obtain a negative direction for Q and consequently a lower bound for the Morse Index of u_ε. Taking $m\to\infty$ we conclude the proof of Theorem 2.

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O. Agudelo - NTIS, Department of Mathematics, Zapadoceska Univerzita v Plzni, Plzen, Czech Republic.
E-mail address: `oiagudel@ntis.zcu.cz`

M. Del Pino - Departamento de Ingeniería Matemática and CMM, Universidad de Chile, Casilla 170 Correo 3, Santiago, Chile.
E-mail address: `delpino@dim.uchile.cl`

J. Wei - Department of Mathematics, University of British Columbia, Vancouver BC V6T 1Z2, Canada and Department of Mathematics, Chinese University of Hong Kong, Shatin, NT, Hong Kong
E-mail address: `wei@math.cuhk.hk`