ON SPIKE SOLUTIONS FOR A SINGULARLY PERTURBED PROBLEM IN A COMPACT RIEMANNIAN MANIFOLD

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Abstract: Let (M, g) be a smooth compact riemannian manifold of dimension $N \geq 2$. We are concerned with the following elliptic problem

$$-\varepsilon^2 \Delta_g u + u = u^{p-1}, \quad u > 0, \qquad in \;\; M.$$

where Δ_g is the Laplace-Beltrami operator on M, $p > 2$ if $N = 2$ and $2 < p < \frac{2N}{N-2}$ if $N \geq 3$, ε is a small real parameter. We prove that there exist a function Φ such that if ξ_0 is a stable critical point of $\Phi(\xi)$ there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, problem (1) has a solution u_ε which concentrates at ξ_0 as ε tends to zero. This result generalizes previous works which handle the case where the scalar curvature function of (M, g) has non-degenerate critical points.

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1. Introduction

Consider the following problem

$$-\varepsilon^2 \Delta_g u + u - u^{p-1} = 0 \qquad \text{in} \;\; M \tag{1}$$

where (M, g) is a smooth compact riemannian manifold without boundary, of dimension $N \geq 2$, $\varepsilon > 0$ is a small parameter and $p > 2$ if $N = 2$, $2 < p < 2^* = \frac{2N}{N-2}$ if $N \geq 3$. The energy functional J_ε associated to (1) is defined by

$$J_\varepsilon[u] = \int_M \left(\frac{\varepsilon^2}{2} |\nabla_g u|^2 + \frac{1}{2} u^2 - \frac{1}{p} u^p\right) d\mu_g \qquad \text{for } u \in H^1(M).$$

There is an extensive litterature regarding this problem in bounded domains of $\mathbb{R}^n$ (with Neumann boundary condition). Solutions concentrating at points or in general positive dimensional manifolds has been found for $2 < p \leq \frac{2(n-k)}{n-k-2}$ for $n - k \geq 3$ and

$p > 2$ for $n - k = 2$ (where k is dimension of the concentration manifold). In fact, when problem (1) is replaced by the following problem

$$\begin{cases} -\varepsilon^2 \Delta u + u - u^{p-1} = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{2}$$

which arises in many branches of the applied sciences. For example, it can be viewed as a steady-state equation for the shadow system of the Gierer-Meinhardt system in biological pattern formation or of parabolic equations in chemotaxis, population dynamics and phase transitions. Many works has been devoted to study spike and bubble solutions. We refer the reader to the pioneering papers, Lin and al. [13, 18, 19], who established the existence of least-energy solutions to (2) and show that for ε small enough the least energy solution has a boundary spike, which approaches the maximum point of the mean curvature H of $\partial\Omega$, as ε goes to zero. Later, in [3, 21] it has been proved that for any stable critical point of H one can construct single boundary spike layer, while in [8, 12, 23] the authors construct multiple boundary spike layer solutions at multiple stable critical points of H. Multiple peak solutions has been also constructed, see [9] who proved that for any integer k there exist a boundary k-peak solutions, whose peaks which collapse to a local minimum point of H. We also mention the papers [1, 2, 7, 6, 10, 22] where interior spike layer solutions are constructed.

As a summary, we observe that since the equation is autonomous is the geometry of the domain who plays a crucial role in the location of point (or submanifold) concentration. This can be seen also once expanding the energy of a single boundary spike solution u_ε, see Ni and Takagi [13, 18]. They proved that the following asymptotic expansion holds

$$J_\varepsilon[u_\varepsilon] = \varepsilon^N \big[\frac{1}{2} I[w] - c_1 \varepsilon H(P_\varepsilon) + o(\varepsilon)\big], \tag{3}$$

where $c_1 > 0$ is a generic constant, P_ε is the unique local maximum point of u_ε and $H(P_\varepsilon)$ is the boundary mean curvature function at $P_\varepsilon \in \partial\Omega$ and where w is the unique ground state solution

$$\begin{cases} \Delta w - w + w^{p-1} = 0, \qquad w > 0 & \text{in } \mathbb{R}^N \\ w(0) = \max_{y \in \mathbb{R}^N} w(y), \qquad \lim_{|y| \to +\infty} w(y) = 0 \end{cases} \tag{4}$$

and $I[w]$ is the ground-state energy

$$I[w] = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla w|^2 dy + \frac{1}{2} \int_{\mathbb{R}^N} w^2 dy - \frac{1}{p} \int_{\mathbb{R}^N} w^p dy.$$

In the Riemannian setting, it turns out that is the scalar curvature function which is relevant for point concentration in M for problem (1) or some of its variants. For example, if one consider the following asymptotically critical elliptic problem

$$-\Delta_g u + h(x)u = u^{2^*-1-\varepsilon}, \qquad u > 0 \quad \text{in } M,$$

where Δ_g stands for the Laplace-Beltrami operator on M, $h(x)$ is a C^1 function on M, $2^* = \frac{2N}{N-2}$ denotes the Sobolev critical exponent, ε is a small real parameter. Micheletti, Pistoia and Vétois [16] proved the existence of blowing-up solutions for this equation in any compact manifold with dimension $N \geq 6$. In particular they prove that in the slightly subcritical case $p_\varepsilon = \frac{N+2}{N-2} - \varepsilon$ blowing-up solutions at some point ξ_0 exist only for large potentials with respect to the potential of the Yamabe equation: $h(\xi_0) > \frac{N-2}{4(N-1)}\mathrm{Scal}_g(\xi_0)$, while in the slightly super-critical case $p_\varepsilon = \frac{N+2}{N-2} + \varepsilon$ blowing-up solutions exist only for small potentials with respect to the potential of the Yamabe equation: $h(\xi_0) < \frac{N-2}{4(N-1)}\mathrm{Scal}_g(\xi_0)$. Using a Lyapunov-Schmidt reduction procedure S. Deng [4] proved that this problem admits a m-peaks solution for any positive integer $m \geq 2$, which blow up and concentrate at some points in M. Esposito and Pistoia [5] obtained the existence of bubbling solutions for the problem when $h = \frac{N-2}{4(N-1)}\mathrm{Scal}_g$, the solutions blow-up at a maximum point of the Weyl curvature tensor of g.

We refer also to [14, 15] where it has been proven that given any non-denegenerate critical point p_0 of scalar curvature function $\mathbf{s}$ then problem (1) possesses solutions which concentrate near p_0. In this short note, we are interested in the cases that are not covered by these results, namely the cases where the scalar curvature has degenerate critical points. This is for example the case when $(M; g)$ is an Einstein manifold or more generally when the scalar curvature of g is constant. Similar result have been proven in a different setting: families of constant mean curvature hypersurfaces foliating geodesic spheres has been obtained by Pacard and Xu [20] extending previous works of R. Ye [25, 26]. In fact, R. Ye proved the existence of branches of constant mean curvature hypersurfaces each of which is associated to non-degenerate critical points of the scalar curvature. Moreover, he proved that the elements of these branches form a local foliation of a neighborhood of p by constant mean curvature hypersurfaces. Pacard and Xu relaxed the condition on the point concentration, showing that the same construction hold provided p is a critical point of some function $\phi:\ M \times (0, \rho_0) \to \mathbb{R}$ satisfying

$$\|\phi(\rho, \cdot) - \mathbf{s} - \rho^2 \mathbf{r}\|_{C^k(M)} \leq C_k\, \rho^3$$

where $\mathbf{s}$ is the scalar curvature function and $\mathbf{r}$ is an explicit function depending on the dimension m, the scalar curvature function, the Ricci tensor, $\mathrm{Ric_q}$, and the riemannian tensor and R_q at the point q. Our aim is then, to prove similar result for problem (1) relaxing the results in [14, 15, 16]. To this purpose, we first define

$$\Phi(\xi) = \mathrm{Scal}_g(\xi) - \frac{1}{120(N+2)}\, \varepsilon^2\, \Xi(\xi), \tag{5}$$

where $\mathrm{Scal}_g(\xi)$ denotes the scalar curvature of g at ξ, and

$$\Xi(\xi) = -18\Delta_g \mathrm{Scal}_g(\xi) + 8\|\mathrm{Ric}_\xi\|^2 + 5\mathrm{Scal_g}(\xi)^2 - 3\|\mathrm{R}_\xi\|^2 \tag{6}$$

with Ric_ξ denotes the Ricci tensor and R_ξ is the riemannian tensor at the point ξ. Our main result is the following

Theorem 1.1. *There exist $\varepsilon_0 > 0$ and a smooth function $\phi : M \times (0, \varepsilon_0) \to \mathbb{R}$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and for any critical point ξ_0 of the function $\phi(\varepsilon, \cdot)$, problem (1) has a solution u_ε which concentrates at the point ξ_0 as ε tends to zero. Moreover ϕ satisfies*

$$\|\phi - Scal_g(\xi) - \frac{1}{120(N+2)}\Xi(\xi)\varepsilon^2\|_{C^k} \le C\varepsilon^3$$

where Ξ is given in the above formula (6).

The first main step in proving our main result is a high order expansions of the metric near geodesic normal coordinates. This is done in Section 2 together with the expansion of the volume element and some preliminary results. In Section 3 we give the existence result. Section 4 will be devoted to the finite dimensional reduction prodecure while in Section 5 (Appendix) we prove Proposition 3.2 as well as some preliminary lemmas.

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2. Some preliminary results

In this subsection we first introduce Fermi coordinates (geodesic normal coordinates) in a neighborhood of a point $\xi \in M$. Let E_i, $i = 1, \dots, N$, be an orthonormal basis of $T_\xi M$. We next, choose normal geodesic coordinates in a neighborhood of ξ in M through the map

$$F(z) := \exp_\xi(z_i\, E_i), \qquad z := (z_1, \dots, z_N),$$

where $\exp_\xi$ is the exponential map on M at ξ and where we have used Einstein's convention of summation over repeated indices. This yields the coordinate vector fields $X_i := F_*(\partial_{z^i})$. Recall that the Fermi coordinates above are defined so that the metric coefficients $g_{ij} = g(X_i, X_j) = \delta_{ij}$ at ξ and $X_k g_{ij} = X_k g(X_i, X_j) = 0$ at ξ. We now compute higher terms in the Taylor expansions of the functions g_{ij}. The metric coefficients at $q := F(z)$ are given in terms of geometric data at $\xi := F(0)$ and $|z| := \left(z_1^2 + \ldots + z_N^2\right)^{1/2}$.

We now give the well known expansion for the metric in normal coordinates, we refer the reader to [11, 20] and some references therein for the expansion of the metric coefficients. The expansions of the inverse of the metric and the volume element follows then from classical taylor expansions.

Lemma 2.1. *In a normal coordinates neighborhood of $\xi \in M$, the Taylor series of g around ξ is given by*

$$g_{ij} = \delta_{ij} + \frac{1}{3} R_{kijl} z_k z_l + \frac{1}{6} \nabla_m R_{kijl} z_k z_l z_m + \Big(\frac{1}{20} \nabla_{pq} R_{kijl} + \frac{2}{45} R_{kilr} R_{pjqr}\Big) z_k z_l z_p z_q + \mathcal{O}(|z|^5),$$

as $|z| \to 0$. Moreover,

$$\begin{aligned} g^{ij} = \delta_{ij} - \frac{1}{3} R_{kijl} z_k z_l - \frac{1}{6} \nabla_m R_{kijl} z_k z_l z_m - \Big(\frac{1}{20} \nabla_{pq} R_{kijl} + \frac{2}{45} R_{kilr} R_{pjqr}\Big) z_k z_l z_p z_q \\ + \frac{1}{9} R_{kisl} R_{psjq} z_k z_l z_p z_q + \mathcal{O}(|z|^5). \end{aligned}$$

Furthermore, the volume element on normal coordinates has the following expansion

$$\begin{aligned} \sqrt{det(g)} = 1 - \frac{1}{6} R_{kl} z_k z_l - \frac{1}{12} \nabla_m R_{kl} z_k z_l z_m \\ - \Big(\frac{1}{40} \nabla_{pq} R_{kl} + \frac{1}{180} R_{kilr} R_{piqr} - \frac{1}{72} R_{kl} R_{pq}\Big) z_k z_l z_p z_q + \mathcal{O}(|z|^5). \end{aligned}$$

Here all curvature terms are evaluated at ξ. Convention ever repeated indices is understood and where the symbol $\mathcal{O}(|z|^r)$ indicates an analytic function such that it and its partial derivatives of any order, with respect to the vector fields $z_j X_i$, are bounded by a constant times $|z|^r$ in some fixed neighborhood of 0.

Let H_ε be the Hilbert space $H_g^1(M)$ equipped with the inner product

$$\langle u, v \rangle_\varepsilon := \frac{1}{\varepsilon^N} \left(\varepsilon^2 \int_M \nabla_g u \nabla_g v d\mu_g + \int_M u v d\mu_g \right),$$

which induces the norm

$$\|u\|_\varepsilon^2 := \frac{1}{\varepsilon^N} \left(\varepsilon^2 \int_M |\nabla_g u|^2 d\mu_g + \int_M u^2 d\mu_g \right),$$

Let L_ε^q be the Banach space $L_g^q(M)$ with the norm

$$|u|_{q,\varepsilon} = \left(\frac{1}{\varepsilon^N} \int_M |u|^q \, d\mu_g \right)^{1/q}.$$

It is clear that the embedding $i_\varepsilon : H_\varepsilon \hookrightarrow L^p_\varepsilon$ is a compact continuous map. We let $i^*_\varepsilon : L^{p'}_\varepsilon \hookrightarrow H_\varepsilon$ be the adjoint operator of the embedding i_ε, where $p' = \frac{p}{p-1}$, the embedding i^*_ε is a continuous map such that for any w in $L^{p'}_\varepsilon$, the function $u = i^*_\varepsilon(w)$ in H_ε is the unique solution of the equation $-\varepsilon^2 \Delta_g u + u = w$ in M, that is

$$\langle i^*_\varepsilon(w), \phi\rangle_\varepsilon = \frac{1}{\varepsilon^N}\int_M w\phi d\mu_g, \quad \phi \in H_\varepsilon.$$

By the continuity of the embedding H_ε into L^p_ε, we have for any $w \in L^{p'}_\varepsilon$, one has

$$\|i^*_\varepsilon(w)\|_\varepsilon \le C|w|_{p',\varepsilon} \tag{7}$$

for some positive constant C independent of w.

We can rewrite problem (1) in the equivalent way

$$u = i^*_\varepsilon(f(u)), \quad u \in H_\varepsilon, \tag{8}$$

where $f(u) = (u^+)^{p-1}$.

Next, we introduce the following equation which correspond to limiting equation to problem (1). It is well know that there exists a unique positive spherically symmetric function $U \in H^1(\mathbb{R}^N)$ such that

$$-\Delta U + U = U^{p-1}, \qquad in \ \ \mathbb{R}^N. \tag{9}$$

Moreover, the function U and its derivatives are exponentially decaying at infinity, namely

$$\lim_{|z|\to\infty} U(|z|)|z|^{\frac{N-1}{2}} e^{|z|} = c > 0, \qquad \lim_{|z|\to\infty} U'(|z|)|z|^{\frac{N-1}{2}} e^{|z|} = -c. \tag{10}$$

Let us define a smooth cut-off function χ_r satisfies

$$\chi_r(z) := \begin{cases} 1 & \text{if} \quad z \in B(0, \frac{r}{2}); \\ \in (0,1) & \text{if} \quad z \in B(0,r)\backslash B(0,\frac{r}{2}); \\ 0 & \text{if} \quad z \in \mathbb{R}^n \backslash B(0,r), \end{cases} \tag{11}$$

and $|\nabla \chi_r(z)| \le \frac{2}{r}$, $|\nabla^2 \chi_r(z)| \le \frac{2}{r^2}$. For any point ξ in M and for any positive real number λ, we define the function $W_{\lambda,\xi}$ on M by

$$W_{\varepsilon,\xi}(x) := \begin{cases} \chi_r\left(\exp_\xi^{-1}(x)\right) U_\varepsilon\left(\exp_\xi^{-1}(x)\right) & \text{if } x \in B_g(\xi, r); \\ 0 & \text{otherwise}, \end{cases} \tag{12}$$

where $U_\varepsilon(z) = U(\frac{z}{\varepsilon})$.

We will look for a solution to (8) or equivalently to (1) as $u_\varepsilon := W_{\varepsilon,\xi} + \phi$, where the rest term ϕ belongs to a suitable space which will be introduced in the following.

It is well known that every solution to the linear equation

$$-\Delta\psi + \psi = (p-1)U^{p-2}\psi, \qquad \text{in} \ \ \mathbb{R}^N \tag{13}$$

is a linear combination of the functions

$$\psi^i(z) = \frac{\partial U(z)}{\partial z_i}, \quad i = 1, 2, ..., N.$$

Let us define on M the functions

$$Z^i_{\varepsilon,\xi}(x) := \begin{cases} \chi_r \left(\exp_\xi^{-1}(x)\right) \psi^i_\varepsilon \left(\exp_\xi^{-1}(x)\right) & \text{if } x \in B_g(\xi, r); \\ 0 & \text{otherwise,} \end{cases} \tag{14}$$

where $\psi^i_\varepsilon = \psi^i(\frac{z}{\varepsilon})$.

Let

$$K_{\varepsilon,\xi} = \text{Span}\left\{Z^i_{\varepsilon,\xi} : \ i = 1, 2, \cdots, N\right\},$$

and

$$K^\perp_{\varepsilon,\xi} = \left\{\phi \in H_\varepsilon : \ \langle \phi, Z^i_{\varepsilon,\xi}\rangle_\varepsilon = 0, \ \forall \ i = 1, 2, \cdots, N\right\}.$$

We will look for a solution to (8), or equivalently to (1), of the form

$$u_\varepsilon = W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi} \tag{15}$$

where the rest term $\phi_{\varepsilon,\xi}$ belongs to the space $K^\perp_{\varepsilon,\xi}$ and the functions $W_{\varepsilon,\xi}$ are defined in (12) .

Let $\Pi_{\varepsilon,\xi} : H_\varepsilon \to K_{\varepsilon,\xi}$ and $\Pi^\perp_{\varepsilon,\xi} : H_\varepsilon \to K^\perp_{\varepsilon,\xi}$ be the orthogonal projections. In order to solve problem (8) we will solve the system

$$\Pi^\perp_{\varepsilon,\xi}\left\{W_{\varepsilon,\xi} + \phi - i^*_\varepsilon\left[f\left(W_{\varepsilon,\xi} + \phi\right)\right]\right\} = 0, \tag{16}$$

$$\Pi_{\varepsilon,\xi}\left\{W_{\varepsilon,\xi} + \phi - i^*_\varepsilon\left[f\left(W_{\varepsilon,\xi} + \phi\right)\right]\right\} = 0. \tag{17}$$

3. The existence result

Let us introduce the linear operator $L_{\varepsilon,\xi} : H_\varepsilon \cap K^\perp_{\varepsilon,\xi} \to K^\perp_{\varepsilon,\xi}$ defined by

$$L_{\varepsilon,\xi}(\phi) := \Pi^\perp_{\varepsilon,\xi}\left\{\phi - i^*_\varepsilon\left[f(W_{\varepsilon,\xi})\phi\right]\right\}. \tag{18}$$

This operator is well defined by using (7). Therefore equation (16) is equivalent to

$$L_{\varepsilon,\xi}(\phi) = N_{\varepsilon,\xi}(\phi) + R_{\varepsilon,\xi} \tag{19}$$

where

$$N_{\varepsilon,\xi}(\phi) = \Pi^\perp_{\varepsilon,\xi}\left\{i^*_\varepsilon\left[f(W_{\varepsilon,\xi} + \phi) - f(W_{\varepsilon,\xi}) - f'(W_{\varepsilon,\xi})\phi\right]\right\}, \tag{20}$$

and

$$R_{\varepsilon,\xi} = \Pi^\perp_{\varepsilon,\xi}\left\{i^*_\varepsilon\left(f(W_{\varepsilon,\xi})\right) - W_{\varepsilon,\xi}\right\}. \tag{21}$$

We first give the following result whose proof is postponed until Section 4 to solve equation (16).

Proposition 3.1. *There exists $\varepsilon_0 > 0$ and $C > 0$ such that for any $\xi \in M$ and for any $\varepsilon \in (0, \varepsilon_0)$, there exists a unique $\phi_{\varepsilon,\xi} = \phi(\varepsilon, \xi)$ which solves equation (16), which is continuously differential with respect to ξ, moreover,*

$$\|\phi_{\varepsilon,\xi}\|_\varepsilon \leq C\varepsilon^2. \tag{22}$$

We now introduce the functional $J_\varepsilon : H_\varepsilon \to \mathbb{R}$ defined by

$$J_\varepsilon(u) = \frac{1}{\varepsilon^N}\left(\frac{1}{2}\int_M \varepsilon^2|\nabla_g u|^2\, d\mu_g + \frac{1}{2}\int_M u^2\, d\mu_g - \frac{1}{p}\int_{\mathcal{M}} u_+^p\, d\mu_g\right),$$

It is well known that any critical point of J_ε is solution to problem (1). We also define the functional $\widetilde{J}_\varepsilon : M \to \mathbb{R}$ by

$$\widetilde{J}_\varepsilon(\xi) = J_\varepsilon\left(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}\right), \tag{23}$$

where $W_{\varepsilon,\xi}$ is as (12) and $\phi_{\varepsilon,\xi}$ is given by Proposition 3.1.

Next, we prove that the criticl points of $\widetilde{J}_\varepsilon$ are the solutions to problem (17).

Proposition 3.2. *For ε small, if ξ is a critical point of the functional $\widetilde{J}_\varepsilon$, then $W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}$ is a solution of (8), or equivalently of problem (1).*

Proof. We argue as in Lemma 4.1 in [14]. □

The problem is thus reduced to finding critical points of $\widetilde{J}_\varepsilon$ and so it is necessary to compute the asymptotic expansion of $\widetilde{J}_\varepsilon$.

Proposition 3.3. *(i) For $\varepsilon > 0$ small enough, one has*

$$\widetilde{J}_\varepsilon(\xi) = J_\varepsilon\left(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}\right) = J_\varepsilon\left(W_{\varepsilon,\xi}\right) + o(\varepsilon^4), \tag{24}$$

uniformly with respect to ξ.

(ii) It holds that

$$J_\varepsilon(W_{\varepsilon,\xi}(x)) = c_0 - c_1\Phi(\xi)\varepsilon^2 + o(\varepsilon^4), \tag{25}$$

where $\Phi(\xi)$ is defined in (5), and

$$c_0 = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla U|^2 + U^2)dz - \frac{1}{p}\int_{\mathbb{R}^N} U^p dz,$$
$$c_1 = \frac{1}{6}\int_{\mathbb{R}^N}\left(\frac{U'(|z|)}{|z|}\right)^2 z_1^4 dz.$$

Proof. The proof is postponed to Appendix. □

Proof of Theorem 1.1: By the assumption that the function $\mathrm{Scal}_g(\xi) - c_2\Xi(\xi)\varepsilon^2$ exists stable critical point ξ_0. Theorem 1.1 follows from Proposition 3.2 and 3.3.

4. The finite dimensional reduction

The section is devoted to the proof of Proposition 3.1. Let us recall that the linear operator $L_{\varepsilon,\xi}$ which is given in (18). As a first step, we want to study the invertibility of $L_{\varepsilon,\xi}$.

Lemma 4.1. *For any $\phi \in H_\varepsilon \cap K^{\perp}_{\varepsilon,\xi}$ and $\xi \in M$, if ε is small enough, there holds*

$$\|L_{\varepsilon,\xi}(\phi)\|_\varepsilon \geq C\|\phi\|_\varepsilon, \tag{26}$$

where C is a positive constant.

Proof. The proof follows that of Proposition 3.1 in [14], but we give the sketch of proof for completeness. By contradiction, we assume that there esist sequences $\varepsilon_n \to 0$, $\xi_n \in M$ such that (up to a subsequence) $\xi_n \to \xi$, $\phi_n \in K^{\perp}_{\varepsilon_n,\xi_n}$, with $\|\phi_n\|_{\varepsilon_n} = 1$ such that

$$L_{\varepsilon_n,\xi_n}(\phi_n) = \psi_n \quad \text{ and } \quad \|\psi_n\|_{\varepsilon_n} \to 0.$$

Thus, there exists $\zeta_n \in K_{\varepsilon_n,\xi_n}$ such that

$$\phi_n - i^*_{\varepsilon_n}[f'(W_{\varepsilon_n,\xi_n})\phi_n] = \psi_n + \zeta_n. \tag{27}$$

We will divide into the following three steps to get a contradiction.

Step 1. We claim that

$$\|\zeta_n\|_{\varepsilon_n} \longrightarrow 0 \quad as \quad n \to \infty. \tag{28}$$

In fact, let $\zeta_n := \sum\limits_{k=1}^{N} a^k_n Z^k_{\varepsilon_n,\xi_n}$. For any $h = 1, \cdots, N$, we multiply (27) by $Z^h_{\varepsilon_n,\xi_n}$, and since $\phi_n, \psi_n \in K^{\perp}_{\varepsilon_n,\xi_n}$, we find

$$\sum_{k=1}^{N} a^k_n \langle Z^k_{\varepsilon_n,\xi_n}, Z^h_{\varepsilon_n,\xi_n}\rangle_{\varepsilon_n} = \langle i^*_{\varepsilon_n}[f'(W_{\varepsilon_n,\xi_n})\phi_n], Z^h_{\varepsilon_n,\xi_n}\rangle_{\varepsilon_n} = \frac{1}{\varepsilon_n^N}\int_M f'(W_{\varepsilon_n.\xi_n})\phi_n Z^h_{\varepsilon_n,\xi_n} \tag{29}$$

By a direct computations, we have

$$\langle Z^k_{\varepsilon_n,\xi_n}, Z^h_{\varepsilon_n,\xi_n}\rangle_{\varepsilon_n} = \begin{cases} c + o(1) & \text{if } h = k; \\ o(1) & \text{if } h \neq k, \end{cases} \tag{30}$$

where c is a positive constant. Moreover, set

$$\tilde{\phi}_n(x) = \begin{cases} \phi_n(\exp_n(\varepsilon_n z)\chi_r(\varepsilon_n z)) & \text{if } z \in B(0, r/\varepsilon_n); \\ 0 & \text{otherwise,} \end{cases}$$

then we have that $\|\tilde{\phi}_n\|_{H^1(\mathbb{R}^N)} \leq C\|\tilde{\phi}_n\|_{\varepsilon_n} \leq C$. Therefore, we can assume that $\tilde{\phi}_n$ converges to some $\tilde{\phi}$ weakly in $H^1(\mathbb{R}^N)$ and strongly in $L^q_{loc}(H^1(\mathbb{R}^N)$ for any $q \in [2, 2^*)$ if $N \geq 3$ or $q \geq 2$ if $N = 2$. Since $\phi_n \in K^{\perp}_{\varepsilon_n,\xi_n}$, we have

$$\begin{aligned}
&-\frac{1}{\varepsilon_n^N}\int_M f'(W_{\varepsilon_n.\xi_n})\phi_n Z^h_{\varepsilon_n,\xi_n} d\mu_g \\
&= \frac{1}{\varepsilon_n^N}\int_M [\varepsilon_n^2 \nabla_g Z^h_{\varepsilon_n,\xi_n}\nabla_g \phi_n + Z^h_{\varepsilon_n,\xi_n}\phi_n - f'(W_{\varepsilon_n.\xi_n})\phi_n Z^h_{\varepsilon_n,\xi_n}] d\mu_g \\
&= \int_{\mathbb{R}^N} (\nabla \psi^h \nabla \tilde{\phi} + \psi^h \tilde{\phi} - f'(U)\psi^h \tilde{\phi}) dz + o(1) = o(1). \qquad (31)
\end{aligned}$$

From (29)-(31), we get that $a_n^k \to 0$ for any $k = 1, \cdots, N$, and then (28) follows.

Step 2. Let us write $u_n := \phi_n - \psi_n - \zeta_n$, there holds

$$\frac{1}{\varepsilon_n^N}\int_M f'(W_{\varepsilon_n,\xi_n})u_n^2 \to 1 \quad as \quad n \to \infty. \qquad (32)$$

In fact, since $\|\phi_n\|_{\varepsilon_n} = 1$, $\|\psi_n\|_{\varepsilon_n} \to 0$, and (28), we have that

$$\|u_n\|_{\varepsilon_n} \to 1. \qquad (33)$$

Moreover, u_n satisfies the following equation

$$-\varepsilon_n^2 \Delta_g u_n + u_n = f'(W_{\varepsilon_n,\xi_n})u_n + f'(W_{\varepsilon_n,\xi_n})(\psi_n + \zeta_n) \quad \text{in} \quad M. \qquad (34)$$

Multiplying above equation by u_n, we deduce that

$$\|u_n\|^2_{\varepsilon_n} = \frac{1}{\varepsilon_n^N}\int_M f'(W_{\varepsilon_n,\xi_n})u_n^2 + \frac{1}{\varepsilon_n^N}\int_M f'(W_{\varepsilon_n,\xi_n})(\psi_n + \zeta_n)u_n \qquad (35)$$

By Hölder's inequality and Sobolev embedding, we can find

$$\left|\frac{1}{\varepsilon_n^N}\int_M f'(W_{\varepsilon_n,\xi_n})(\psi_n + \zeta_n)u_n\right| = o(1). \qquad (36)$$

Then we get (32).

Step 3. Get a contradiction.

Indeed, we observe that u_n is compactly supported in $B_g(\xi_n, r)$. Set

$$\tilde{u}_n = u_n(\exp_n(\varepsilon_n z)\chi_r(\varepsilon_n z)) \qquad z \in B(0, r/\varepsilon_n).$$

We note that $\|\tilde{u}_n\|^2_{H^1(\mathbb{R}^N)} \leq C\|\tilde{u}_n\|^2_{\varepsilon_n} \leq C$. Then, up to a subsequence, $\tilde{u}_n \to \tilde{u}$ weakly in $H^1(\mathbb{R}^N)$ and strongly in $L^q_{loc}(\mathbb{R}^N)$ for any $q \in [2, 2^*)$ if $N \geq 3$ or $q \geq 2$ if $N = 2$. By (34) we can get that $\tilde{u}$ solves the problem

$$-\Delta \tilde{u} + \tilde{u} = f'(U)\tilde{u} \quad \text{in} \quad \mathbb{R}^N. \qquad (37)$$

Since $\phi_n, \psi_n \in K^{\perp}_{\varepsilon_n,\xi_n}$, we find

$$\langle Z^h_{\varepsilon_n,\xi_n}, u_n\rangle_{\varepsilon_n} = -\langle Z^h_{\varepsilon_n,\xi_n}, \zeta_n\rangle_{\varepsilon_n} \le \|Z^h_{\varepsilon_n,\xi_n}\|_{\varepsilon_n}\|\zeta_n\|_{\varepsilon_n} = o(1). \tag{38}$$

On the other hand, we have

$$\begin{aligned}\langle Z^h_{\varepsilon_n,\xi_n}, u_n\rangle_{\varepsilon_n} &= \frac{1}{\varepsilon_n^N}\int_M [\varepsilon_n^2 \nabla_g Z^h_{\varepsilon_n,\xi_n}\nabla_g u_n + Z^h_{\varepsilon_n,\xi_n} u_n] d\mu_g \\ &= \int_{B(0,r/\varepsilon_n)} \Bigg[\sum_{i,j=1}^N g^{ij}_{\xi_n}(\varepsilon_n z)\frac{\partial}{\partial z_i}(\psi^h(z)\chi_r(\varepsilon_n z))\frac{\partial}{\partial z_i}(\tilde u_n(z)) \\ &\qquad + \psi^h(z)\chi_r(\varepsilon_n z)\tilde u_n(z) u_n\Bigg]|g_{\xi_n}(\varepsilon_n z)|^{\frac12} dz \\ &= \int_{\mathbb{R}^N} [\nabla\psi^h\nabla\tilde u + \psi^h\tilde u] dz + o(1). \end{aligned}\tag{39}$$

From (38) and (39) we obtain that

$$\int_{\mathbb{R}^N} [\nabla\psi^h\nabla\tilde u + \psi^h\tilde u] dz = 0, \quad h = 1, \cdots, N. \tag{40}$$

Therefore, by (37) and (40) we get that $\tilde u_n \to 0$ weakly in $H^1(\mathbb{R}^N)$ and strongly in $L^q_{loc}(\mathbb{R}^N)$ for any $q \in [2, 2^*)$ if $N \ge 3$ or $q \ge 2$ if $N = 2$. Then

$$\begin{aligned}\frac{1}{\varepsilon_n^N}\int_M f'(W_{\varepsilon_n,\xi_n})u_n^2 &\le \frac{1}{\varepsilon_n^N}\int_{B_g(\xi_n,r)} f'(U_{\varepsilon_n})(exp^{-1}_{\xi_n}(x))u_n^2(x)dx \\ &\le C\int_{B(0,r/\varepsilon_n)} f'(U(z))\tilde u_n^2(z)dz = o(1)\end{aligned}$$

which gives a contradiction because of (32). □

Next, we have the following estimate of $R_{\varepsilon,\xi}$.

Lemma 4.2. *For any $\xi \in M$, if ε is small enough, there holds*

$$\|R_{\varepsilon,\xi}(\phi)\|_\varepsilon \le C\varepsilon^2, \tag{41}$$

where C is a positive constant.

Proof. The proof follows that of Lemma 3.3 in [14], we sketch it here for completeness. Let us introduce the function $W_{\varepsilon,\xi}$ defined by $W_{\varepsilon,\xi} := i^*(V_{\varepsilon,\xi})$, that is,

$$-\varepsilon^2\Delta_g W_{\varepsilon,\xi} + W_{\varepsilon,\xi} = V_{\varepsilon,\xi} \quad \text{on} \quad M.$$

We remark that $W_{\varepsilon,\xi}(x) = 0$ if $x \notin B_g(\xi, r)$. Therefore, we have $V_{\varepsilon,\xi}(x) = 0$, if $x \notin B_g(\xi, r)$ and if $x \in B_g(\xi, r)$,

$$V_{\varepsilon,\xi} = -\varepsilon^2\Delta_g(U_\varepsilon\chi_r) + U_\varepsilon\chi_r$$

$$
\begin{aligned}
&= U_\varepsilon^{p-1}(z)\chi_r(z) = \varepsilon^2 U_\varepsilon(z)\Delta\chi_r(z) - 2\varepsilon^2\nabla U_\varepsilon(z)\nabla\chi_r(z) \\
&\quad +\varepsilon^2(g_\xi^{ij}-\delta_{ij})\partial_{ij}(U_\varepsilon\chi_r) - \varepsilon^2 g_\xi^{ij}\Gamma_{ij}^k\partial_k(U_\varepsilon\chi_r)
\end{aligned} \tag{42}
$$

From (7) and (42), we then have

$$
\begin{aligned}
&\left\|i^*\left(f(W_{\varepsilon,\xi})\right) - W_{\varepsilon,\xi}\right\|_\varepsilon = \left\|i^*\left(f(W_{\varepsilon,\xi})\right) - i^*(V_{\varepsilon,\xi})\right\|_\varepsilon \\
&\le C\left|f(W_{\varepsilon,\xi}) - V_{\varepsilon,\xi}\right|_{p',\varepsilon} = C\left(\frac{1}{\varepsilon^N}\int_M \left|f(W_{\varepsilon,\xi}) - V_{\varepsilon,\xi}\right|^{p'}\ d\mu_g\right)^{\frac{1}{p'}} \\
&= C\left(\frac{1}{\varepsilon^N}\int_{B(0,r)} \left(U_\varepsilon^{p-1}(z)\chi_r^{p-1}(z) - V_{\varepsilon,\xi}(\exp_\xi(z))\right)^{p'}|g_\xi(z)|^{\frac{1}{2}}dz\right)^{\frac{1}{p'}} \\
&\le C\left(\frac{1}{\varepsilon^N}\int_{B(0,r)} \left(U_\varepsilon^{p-1}(z)\chi_r^{p-1}(z) - V_{\varepsilon,\xi}(\exp_\xi(z))\right)^{p'}dz\right)^{\frac{1}{p'}} \\
&\le C\left(\frac{1}{\varepsilon^N}\int_{B(0,r)} \left|U_\varepsilon^{p-1}(z)\left(\chi_r^{p-1}(z) - V_{\varepsilon,\xi}(\exp_\xi(z)\right)\right|^{p'}dz\right)^{\frac{1}{p'}} \\
&\quad +C\varepsilon^2\left(\frac{1}{\varepsilon^N}\int_{B(0,r)} U_\varepsilon^{p'}(z)\left|\Delta\chi_r(z)\right|^{p'}dz\right)^{\frac{1}{p'}} \\
&\quad +C\varepsilon^2\left(\frac{1}{\varepsilon^N}\int_{B(0,r)} \left(\nabla U_\varepsilon(z)\nabla\chi_r(z)\right)^{p'}dz\right)^{\frac{1}{p'}} \\
&\quad +C\varepsilon^2\left(\frac{1}{\varepsilon^N}\int_{B(0,r)} \left|(g_\xi^{ij}(\varepsilon z)-\delta_{ij})\partial_{ij}(U_\varepsilon\chi_r)(z)\right|^{p'}dz\right)^{\frac{1}{p'}} \\
&\quad +C\varepsilon^2\left(\frac{1}{\varepsilon^N}\int_{B(0,r)} \left|g_\xi^{ij}\Gamma_{ij}^k(\varepsilon z)\partial_k(U_\varepsilon\chi_r)(z)\right|^{p'}dz\right)^{\frac{1}{p'}} \\
&= o(\varepsilon^2)
\end{aligned} \tag{43}
$$

by using the fact that U and its derivatives decay exponentially, and

$$
g_\xi^{ij}(\varepsilon z) = \delta_{ij} + O(\varepsilon^2|z|^2), \quad \Gamma_{ij}^k(\varepsilon z) = \Gamma_{ij}^k(0) + O(\varepsilon|z|) = O(\varepsilon|z|).
$$

This concludes the proof of Lemma. □

Proof of Proposition 3.1: In order to solve (16) or equivalently equation (19), we need to find a fixed point for the operator $T_{\varepsilon,\xi} : H_\varepsilon \cap K_{\varepsilon,\xi}^\perp \to H_\varepsilon \cap K_{\varepsilon,\xi}^\perp$ defined

$$
T_{\varepsilon,\xi}(\phi) = L_{\varepsilon,\xi}^{-1}\left(N_{\varepsilon,\xi}(\phi) + R_{\varepsilon,\xi}\right),
$$

for ε small and for any $\xi \in \mathcal{M}$. From Lemma 4.1, we have

$$\|T_{\varepsilon,\xi}(\phi)\| \le C(\|N_{\varepsilon,\xi}(\phi)\|_\varepsilon + \|R_{\varepsilon,\xi}\|_\varepsilon)$$

and

$$\|T_{\varepsilon,\xi}(\phi_1) - T_{\varepsilon,\xi}(\phi_2)\| \le C\|N_{\varepsilon,\xi}(\phi_1) - N_{\varepsilon,\xi}(\phi_2)\|_\varepsilon.$$

By (7), we deduce

$$\|N_{\varepsilon,\xi}(\phi)\|_\varepsilon \le C\left|f(W_{\varepsilon,\xi}+\phi) - f(W_{\varepsilon,\xi}) - f'(W_{\varepsilon,\xi})\phi\right|_\varepsilon,$$

and

$$\begin{aligned}
&\|N_{\varepsilon,\xi}(\phi_1) - N_{\varepsilon,\xi}(\phi_2)\|_\varepsilon \le C\left|f(W_{\varepsilon,\xi}+\phi_1) - f(W_{\varepsilon,\xi}+\phi_2) - f'(W_{\varepsilon,\xi})(\phi_1-\phi_2)\right|_\varepsilon \\
&\le C\left|\left(f'(W_{\varepsilon,\xi}+\phi_2+\tau(\phi_1-\phi_2)) - f'(W_{\varepsilon,\xi})\right)(\phi_1-\phi_2)\right|_{p',\varepsilon} \\
&\le C\left|f'(W_{\varepsilon,\xi}+\phi_2+\tau(\phi_1-\phi_2)) - f'(W_{\varepsilon,\xi})\right|_{\frac{p}{p-2},\varepsilon}|\phi_1-\phi_2)|_{p,\varepsilon} \\
&\le C\|\phi_1-\phi_2\|_\varepsilon
\end{aligned}$$

for $\|\phi_1\|_\varepsilon$, $\|\phi_2\|_\varepsilon$ are small enough.

Moreover, we get

$$\|N_{\varepsilon,\xi}(\phi)\|_\varepsilon \le C(\|\phi\|_\varepsilon^{p-1} + \|\phi\|_\varepsilon^2).$$

Thus we obtain that $T_{\varepsilon,\xi}$ is a contraction map on suitable ball of H_ε, which has a fixed point in the centered at o with radius $c\varepsilon^2$ in $K_{\varepsilon,\xi}^\perp$ for a suitable constant c. Furthermore, the map $\xi \to \phi_{\varepsilon,\xi}$ is the C^1-function by the Implict Function Theorem.

5. Appendix: Proof of Proposition 3.2

In this section, we give the proof of Proposition 3.2. We recall the expansions given in Lemma 2.1. To prove Proposition 3.2 we argue as in Lemma 5.1 in [14]. We compute the energy $J_\varepsilon(W_{\varepsilon,\xi})$. We get

$$\begin{aligned}
&\frac{\varepsilon^2}{\varepsilon^N}\frac{1}{2}\int_M |\nabla_g W_{\varepsilon,\xi}(x)|^2 d\mu_g = \frac{1}{2}\int_{B(0,r/\varepsilon)} g_\xi^{ij}(\varepsilon z)\frac{\partial(U(z)\chi_{r/\varepsilon}(z))}{\partial z_i}\frac{\partial(U(z)\chi_{r/\varepsilon}(z))}{\partial z_j}\sqrt{\det(g_\xi(\varepsilon z))}dz \\
&= \frac{1}{2}\int_{B(0,r/\varepsilon)} \frac{\partial(U(z)\chi_{r/\varepsilon}(z))}{\partial z_i}\frac{\partial(U(z)\chi_{r/\varepsilon}(z))}{\partial z_j}\Bigg[\delta_{ij} - \frac{\varepsilon^2}{3}R_{kijl}z^kz^l - \frac{\varepsilon^3}{6}\nabla_m R_{kijl}z^kz^lz^m \\
&\quad -\varepsilon^4\Big(\frac{1}{20}\nabla_{pq}R_{kijl} + \frac{2}{45}R_{kilr}R_{pjqr} - \frac{1}{9}R_{kisl}R_{psjq}\Big)z^kz^lz^pz^q + O(\varepsilon^5|z|^5)\Bigg]\times \\
&\quad \times\Big[1 - \frac{\varepsilon^2}{6}R_{kl}z^kz^l - \frac{\varepsilon^3}{12}\nabla_m R_{kl}z^kz^lz^m - \varepsilon^4\Big(\frac{1}{40}\nabla_{pq}R_{kl} + \frac{1}{180}R_{kilr}R_{piqr}\Big)z^kz^lz^pz^q \\
&\quad + \frac{\varepsilon^4}{72}R_{kl}R_{pq}z^kz^lz^pz^q + O(\varepsilon^5|z|^5)\Big]dz
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{\mathbb{R}^N}|\nabla U|^2dz - \frac{\varepsilon^2}{2}\left[\frac{1}{3}R_{kijl}\int_{\mathbb{R}^N}\Big(\frac{U'(|z|)}{|z|}\Big)^2 z_iz_jz_kz_l dz + \frac{1}{6}R_{kl}\int_{\mathbb{R}^N}|\nabla U|^2 z_kz_l dz\right]\\
&\quad +\frac{\varepsilon^4}{2}\Bigg[-\Big(\frac{1}{20}\nabla_{pq}R_{kijl}+\frac{2}{45}R_{kilr}R_{pjqr}-\frac{1}{9}R_{kisl}R_{psjq}\Big)\int_{\mathbb{R}^N}\Big(\frac{U'(|z|)}{|z|}\Big)^2 z_iz_jz_kz_lz_pz_q dz\\
&\qquad +\frac{1}{18}R_{kijl}R_{pq}\int_{\mathbb{R}^N}\Big(\frac{U'(|z|)}{|z|}\Big)^2 z_iz_jz_kz_lz_pz_q dz\\
&\qquad -\left(\frac{1}{40}R_{kl,pq}+\frac{1}{180}R_{kilr}R_{piqr}-\frac{1}{72}R_{kl}R_{pq}\right)\int_{\mathbb{R}^N}|\nabla U|^2 z_kz_lz_pz_q dz\Bigg] + o(\varepsilon^4).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\frac{1}{\varepsilon^N}\frac{1}{2}\int_M W_{\varepsilon,\xi}(x)^2 d\mu_g\\
&= \frac{1}{2}\int_{B(0,r/\varepsilon)}(U(z)\chi_{r/\varepsilon}(z))^2\sqrt{\det(g_\xi(\varepsilon z))}dz\\
&= \frac{1}{2}\int_{\mathbb{R}^N}U^2dz - \frac{\varepsilon^2}{12}R_{kl}\int_{\mathbb{R}^N}U^2 z_kz_l dz\\
&\quad -\frac{\varepsilon^4}{2}\left(\frac{1}{40}\nabla_{pq}R_{kl}+\frac{1}{180}R_{kilr}R_{piqr}-\frac{1}{72}R_{kl}R_{pq}\right)\int_{\mathbb{R}^N}U^2 z_kz_lz_pz_q dz + o(\varepsilon^4),
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{\varepsilon^N}\frac{1}{p}\int_M W_{\varepsilon,\xi}(x)^p d\mu_g\\
&= \frac{1}{p}\int_{B(0,r/\varepsilon)}(U(z)\chi_{r/\varepsilon}(z))^p\sqrt{\det(g_\xi(\varepsilon z))}dz\\
&= \frac{1}{p}\int_{\mathbb{R}^N}U^pdz - \frac{\varepsilon^2}{6p}R_{kl}\int_{\mathbb{R}^N}U^p z_kz_l dz\\
&\quad -\frac{\varepsilon^4}{p}\left(\frac{1}{40}\nabla_{pq}R_{kl}+\frac{1}{180}R_{kilr}R_{piqr}-\frac{1}{72}R_{kl}R_{pq}\right)\int_{\mathbb{R}^N}U^p z_kz_lz_pz_q dz + o(\varepsilon^4).
\end{aligned}
$$

Then we get

$$
J_\varepsilon(W_{\varepsilon,\xi}(x)) = c_0 + \varepsilon^2\Theta_1(\xi) + \varepsilon^4\Theta_2(\xi) + o(\varepsilon^4), \tag{44}
$$

where

$$
c_0 = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla U|^2+U^2)dz - \frac{1}{p}\int_{\mathbb{R}^N}U^pdz,
$$

$$\begin{aligned}
\Theta_1(\xi) &= -\frac{1}{6}R_{kijl}\int_{\mathbb{R}^N}\Big(\frac{U'(|z|)}{|z|}\Big)^2 z_iz_jz_kz_l dz - \frac{1}{12}R_{kl}\int_{\mathbb{R}^N}|\nabla U|^2 z_kz_l dz \\
&\quad -\frac{1}{12}R_{kl}\int_{\mathbb{R}^N}U^2 z_kz_l dz + \frac{1}{6p}R_{kl}\int_{\mathbb{R}^N}U^p z_kz_l dz \\
&= -\frac{1}{6}R_{kijl}\int_{\mathbb{R}^N}\Big(\frac{U'(|z|)}{|z|}\Big)^2 z_iz_jz_kz_l dz \\
&\quad -\frac{1}{6}R_{kk}\Big[\frac{1}{2}\int_{\mathbb{R}^N}|\nabla U|^2 z_k^2 dz + \frac{1}{2}\int_{\mathbb{R}^N}U^2 z_k^2 dz - \frac{1}{p}\int_{\mathbb{R}^N}U^p z_k^2 dz\Big] \\
&= -\frac{1}{6}R_{kijl}\int_{\mathbb{R}^N}\Big(\frac{U'(|z|)}{|z|}\Big)^2 z_iz_jz_kz_l dz - \frac{1}{6}R_{kk}\int_{\mathbb{R}^N}\Big(\frac{\partial U}{\partial z_k}\Big)^2 z_k^2 dz \\
&= -\frac{1}{6}R_{kijl}\int_{\mathbb{R}^N}\Big(\frac{U'(|z|)}{|z|}\Big)^2 z_iz_jz_kz_l dz - \frac{1}{6}R_{kk}\int_{\mathbb{R}^N}\Big(\frac{U'(|z|)}{|z|}\Big)^2 z_1^4 dz \\
&= -\frac{1}{6}\mathrm{Scal}_g(\xi)\int_{\mathbb{R}^N}\Big(\frac{U'(|z|)}{|z|}\Big)^2 z_1^4 dz \\
&= -c_1\,\mathrm{Scal}_g(\xi) \qquad (45)
\end{aligned}$$

where $c_1 = \frac{1}{6}\int_{\mathbb{R}^N}\Big(\frac{U'(|z|)}{|z|}\Big)^2 z_1^4 dz$. Moreover

$$\begin{aligned}
\Theta_2(\xi) &= -\frac{1}{2}\Big(\frac{1}{20}\nabla_{pq}R_{kijl} + \frac{2}{45}R_{kilr}R_{pjqr} - \frac{1}{9}R_{kisl}R_{psjq}\Big)\int_{\mathbb{R}^N}\Big(\frac{U'(|z|)}{|z|}\Big)^2 z_iz_jz_kz_lz_pz_q dz \\
&\quad +\frac{1}{36}R_{kijl}R_{pq}\int_{\mathbb{R}^N}\Big(\frac{U'(|z|)}{|z|}\Big)^2 z_iz_jz_kz_lz_pz_q dz \\
&\quad -\frac{1}{2}\left(\frac{1}{40}\nabla_{pq}R_{kl} + \frac{1}{180}R_{kilr}R_{piqr} - \frac{1}{72}R_{kl}R_{pq}\right)\int_{\mathbb{R}^N}|\nabla U|^2 z_kz_lz_pz_q dz \\
&\quad -\frac{1}{2}\left(\frac{1}{40}\nabla_{pq}R_{kl} + \frac{1}{180}R_{kilr}R_{piqr} - \frac{1}{72}R_{kl}R_{pq}\right)\int_{\mathbb{R}^N}U^2 z_kz_lz_pz_q dz \\
&\quad +\frac{1}{p}\left(\frac{1}{40}\nabla_{pq}R_{kl} + \frac{1}{180}R^s_{kil}R^s_{piq} - \frac{1}{72}R_{kl}R_{pq}\right)\int_{\mathbb{R}^N}U^p z_kz_lz_pz_q dz \\
&= -\frac{1}{2}\Big(\frac{1}{20}\nabla_{pq}R_{kijl} + \frac{2}{45}R_{kilr}R_{pjqr} - \frac{1}{9}R_{kisl}R_{psjq}\Big)\int_{\mathbb{R}^N}\Big(\frac{U'(|z|)}{|z|}\Big)^2 z_iz_jz_kz_lz_pz_q dz \\
&\quad +\frac{1}{36}R_{kijl}R_{pq}\int_{\mathbb{R}^N}\Big(\frac{U'(|z|)}{|z|}\Big)^2 z_iz_jz_kz_lz_pz_q dz
\end{aligned}$$

$$-\left(\frac{1}{40}\nabla_{pq}R_{kl}+\frac{1}{180}R_{kilr}R_{piqr}-\frac{1}{72}R_{kl}R_{pq}\right)\times$$
$$\times\left(\frac{1}{2}\int_{\mathbb{R}^N}|\nabla U|^2 z_kz_lz_pz_qdz+\frac{1}{2}\int_{\mathbb{R}^N}U^2z_kz_lz_pz_qdz-\frac{1}{p}\int_{\mathbb{R}^N}U^pz_kz_lz_pz_qdz\right) \quad (46)$$

Since

$$\nabla_{pq}R_{kijl}\int_{\mathbb{R}^N}\Big(\frac{U'(|z|)}{|z|}\Big)^2 z_iz_jz_kz_lz_pz_qdz=0. \quad (47)$$

here we have used Lemma 5.2. Moreover,

$$R_{kilr}R_{pjqr}\int_{\mathbb{R}^N}\Big(\frac{U'(|z|)}{|z|}\Big)^2 z_iz_jz_kz_lz_pz_qdz=0, \quad (48)$$

$$R_{kisl}R_{psjq}\int_{\mathbb{R}^N}\Big(\frac{U'(|z|)}{|z|}\Big)^2 z_iz_jz_kz_lz_pz_qdz=0, \quad (49)$$

$$R_{kijl}R_{pq}\int_{\mathbb{R}^N}\Big(\frac{U'(|z|)}{|z|}\Big)^2 z_iz_jz_kz_lz_pz_qdz=0. \quad (50)$$

Now, for a radial function $f(r)$, we have

$$\begin{aligned}
&\nabla_{pq}R_{kl}\int_{\mathbb{R}^N}f(r)z_kz_lz_pz_qdz\\
&=\nabla_{kk}R_{kk}\int_{\mathbb{R}^N}f(r)z_1^4dz+\sum_{p\neq q}\nabla_{pp}R_{kk}\int_{\mathbb{R}^N}f(r)z_1^2z_2^2dz+2\sum_{p\neq q}\nabla_{pq}R_{pq}\int_{\mathbb{R}^N}f(r)z_1^2z_2^2dz\\
&=\Big(3\nabla_{kk}R_{kk}+\sum_{p\neq q}\nabla_{pp}R_{kk}+2\sum_{p\neq q}\nabla_{pq}R_{pq}\Big)\int_{\mathbb{R}^N}f(r)z_1^2z_2^2dz\\
&=\left(\big(\nabla_{kk}R_{kk}+\sum_{p\neq q}\nabla_{pp}R_{kk}\big)+2\big(\nabla_{kk}R_{kk}+\sum_{p\neq q}\nabla_{pq}R_{pq}\big)\right)\int_{\mathbb{R}^N}f(r)z_1^2z_2^2dz\\
&=\left(\Delta\mathrm{Scal}_g(\xi)+2\sum_{k,l}\nabla_{kl}R_{kl}\right)\int_{\mathbb{R}^N}f(r)z_1^2z_2^2dz\\
&=\frac{1}{N+2}\left(\Delta\mathrm{Scal}_g(\xi)+2\sum_{k,l}\nabla_{kl}R_{kl}\right)\int_{\mathbb{R}^N}f(r)z_1^2dz,\\
&=\frac{2}{N+2}\Delta\mathrm{Scal}_g(\xi)\int_{\mathbb{R}^N}f(r)z_1^2dz, \qquad (51)
\end{aligned}$$

where in the last equality we have used the fact that

$$\sum_{k,l} 2\nabla_{kl}R_{kl}(\xi) = \sum_{k,l} \nabla_{kk}R_{ll}(\xi) = \Delta_g \mathrm{Scal}_g(\xi)$$

which follows from Bianchi identity.

We next compute the term

$$\begin{aligned}
&R_{kilr}R_{piqr}\int_{\mathbb{R}^N} f(r)z_k z_l z_p z_q dz \\
&= \sum_k R_{kikr}R_{kikr}\int_{\mathbb{R}^N} f(r)z_1^4 dz + \sum_{k\neq p} R_{kikr}R_{pipr}\int_{\mathbb{R}^N} f(r)z_1^2 z_2^2 dz \\
&\quad + \sum_{k\neq l} R_{kilr}R_{kilr}\int_{\mathbb{R}^N} f(r)z_1^2 z_2^2 dz + \sum_{k\neq l} R_{kilr}R_{likr}\int_{\mathbb{R}^N} f(r)z_1^2 z_2^2 dz \\
&= \left(3\sum_k R_{kikr}R_{kikr} + \sum_{k\neq l} R_{kikr}R_{lilr} + \sum_{k\neq l} R_{kilr}R_{kilr} + \sum_{k\neq l} R_{kilr}R_{likr}\right)\int_{\mathbb{R}^N} f(r)z_1^2 z_2^2 dz \\
&= \left(\sum_{k,l} R_{kikr}R_{lilr} + \sum_{k,l} R_{kilr}R_{kilr} + \sum_{k,l} R_{kilr}R_{likr}\right)\int_{\mathbb{R}^N} f(r)z_1^2 z_2^2 dz \\
&= \frac{1}{N+2}\left(\|\mathrm{Ric}_\xi\|^2 + \frac{3}{2}\|R_\xi\|^2\right)\int_{\mathbb{R}^N} f(r)z_1^2 dz. \qquad (52)
\end{aligned}$$

Here we used the fact $R_{kilr}R_{kilr} = 2R_{kilr}R_{krli}$ by the first Bianchi identity. This yields

$$R_{kilr}R_{kilr} + R_{kilr}R_{likr} = 3R_{kilr}R_{krli} = \frac{3}{2}R_{kilr}R_{kilr} = \frac{3}{2}\|R\|^2.$$

Finally

$$\begin{aligned}
&R_{kl}R_{pq}\int_{\mathbb{R}^N} f(r)z_k z_l z_p z_q dz \\
&= \sum_k R_{kk}R_{kk}\int_{\mathbb{R}^N} f(r)z_1^4 dz + \sum_{k\neq p} R_{kk}R_{pp}\int_{\mathbb{R}^N} f(r)z_1^2 z_2^2 dz \\
&\quad + \sum_{k\neq q} R_{kq}R_{kq}\int_{\mathbb{R}^N} f(r)z_1^2 z_2^2 dz + \sum_{k\neq l} R_{kl}R_{kl}\int_{\mathbb{R}^N} f(r)z_1^2 z_2^2 dz \\
&= \left(3\sum_k R_{kk}R_{kk} + \sum_{k\neq p} R_{kk}R_{pp} + 2\sum_{k\neq l} R_{kl}R_{kl}\right)\int_{\mathbb{R}^N} f(r)z_1^2 z_2^2 dz
\end{aligned}$$

$$= \left(\mathrm{Scal}_g(\xi)^2 + 2\sum_{k,l} R_{kl}R_{kl} \right) \int_{\mathbb{R}^N} f(r) z_1^2 z_2^2 dz$$
$$= \frac{1}{N+2}\left(\mathrm{Scal}_g(\xi)^2 + 2\sum_{k,l} R_{kl}R_{kl} \right) \int_{\mathbb{R}^N} f(r) z_1^2 dz$$
$$= \frac{1}{N+2}\left(\mathrm{Scal}_g(\xi)^2 + 2\|\mathrm{Ric}_\xi\|^2 \right) \int_{\mathbb{R}^N} \mathrm{f(r)z_1^2dz}. \tag{53}$$

From (46) to (53), we obtain

$$\Theta_2(\xi) = \frac{1}{2(N+2)}\Bigg[-\frac{1}{20}\Delta \mathrm{Scal}_g(\xi) - \frac{1}{180}\Big(\|\mathrm{Ric}_\xi\|^2 + \frac{3}{2}\|R_\xi\|^2 \Big) + \frac{1}{72}\Big(\mathrm{Scal}_g(\xi)^2 + 2\|\mathrm{Ric}_\xi\|^2 \Big) \Bigg] \times$$
$$\times \left(\frac{1}{2}\int_{\mathbb{R}^N} |\nabla U|^2 z_1^2 dz + \frac{1}{2}\int_{\mathbb{R}^N} U^2 z_1^2 dz - \frac{1}{p}\int_{\mathbb{R}^N} U^p z_1^2 dz \right).$$

Using Lemma 5.1 and Lemma 5.2 below, we find

$$\Theta_2(\xi) = \frac{3}{2(N+2)^2}\Bigg[-\frac{1}{20}\Delta \mathrm{Scal}_g(\xi) - \frac{1}{180}\Big(\|\mathrm{Ric}_\xi\|^2 + \frac{3}{2}\|R_\xi\|^2 \Big) + \frac{1}{72}\Big(\mathrm{Scal}_g(\xi)^2 + 2\|\mathrm{Ric}_\xi\|^2 \Big) \Bigg] \times$$
$$\times \int_{\mathbb{R}^N} \Big(\frac{U'(|z|)}{|z|} \Big)^2 z_1^2 dz$$
$$= \frac{1}{240(N+2)^2}\ \Xi(\xi) \int_{\mathbb{R}^N} \Big(\frac{U'(|z|)}{|z|} \Big)^2 z_1^2 dz$$
$$= \frac{1}{120(N+2)} c_1\ \Xi(\xi), \tag{54}$$

where

$$\Xi(\xi) = -18\Delta_g \mathrm{Scal}_g(\xi) + 8\|\mathrm{Ric}_\xi\|^2 + 5\mathrm{Scal}_g(\xi)^2 - 3\|R_\xi\|^2. \tag{55}$$

Thus, (25) follows from (44), (45), (54) and (55).

Lemma 5.1. *For any $k = 1, 2, \ldots, N$, we have*

$$\frac{1}{2}\int_{\mathbb{R}^N} |\nabla U|^2 z_k^2 dz + \frac{1}{2}\int_{\mathbb{R}^N} U^2 z_k^2 dz - \frac{1}{p}\int_{\mathbb{R}^N} U^p z_k^2 dz = \int_{\mathbb{R}^N} \Big(\frac{\partial U}{\partial z_k} \Big)^2 z_k^2 dz$$
$$= \int_{\mathbb{R}^N} \Big(\frac{U'(|z|)}{|z|} \Big)^2 z_k^4 dz.$$

Proof. Multiply both sides of the equation $-\Delta U + U = U^{p-1}$ in $\mathbb{R}^N$ by $\frac{1}{3} z_k^3 \frac{\partial u}{\partial z_k}$ and then integrate by parts over $\mathbb{R}^N$. □

Lemma 5.2. *It holds that*

$$\alpha := \int_{\mathbb{R}^N} \Big(\frac{U'(|z|)}{|z|}\Big)^2 z_1^4 dz = 3\int_{\mathbb{R}^N} \Big(\frac{U'(|z|)}{|z|}\Big)^2 z_1^2 z_2^2 dz = \frac{3}{N+2}\int_{\mathbb{R}^N} \Big(\frac{U'(|z|)}{|z|}\Big)^2 z_1^2 dz,$$

$$\int_{\mathbb{R}^N} \Big(\frac{U'(|z|)}{|z|}\Big)^2 z_1^2 z_2^2 z_3^2 dz = \frac{1}{N(N+2)}\int_{\mathbb{R}^N} \Big(\frac{U'(|z|)}{|z|}\Big)^2 z_1^2 dz,$$

and

$$\int_{\mathbb{R}^N} \Big(\frac{U'(|z|)}{|z|}\Big)^2 z_i z_j z_k z_l dz = \frac{\alpha}{3}\big(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}\big).$$

Proof. The result follows from the fact that

$$\int_{\mathbb{S}^{N-1}} z_1^4 = 3\int_{\mathbb{S}^{N-1}} z_1^2 z_2^2 = \frac{3}{N+2}\int_{\mathbb{S}^{N-1}} z_1^2,$$

$$\int_{\mathbb{S}^{N-1}} z_1^2 z_2^2 z_3^2 = \frac{1}{N(N+2)}\int_{\mathbb{S}^{N-1}} z_1^2$$

and

$$\int_{\mathbb{S}^{N-1}} z_i z_j z_k z_l = \frac{1}{3}\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{lj} + \delta_{il}\delta_{kj}\right)\int_{\mathbb{S}^{N-1}} z_1^2.$$

This proves the lemma. □

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