Existence and Uniqueness of Large Solutions for a Class of Non Uniformly Elliptic Semilinear Equations

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EXISTENCE AND UNIQUENESS OF LARGE SOLUTIONS FOR A CLASS OF NON UNIFORMLY ELLIPTIC SEMILINEAR EQUATIONS.

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ABSTRACT. In this paper we study existence, uniqueness and asymptotic behavior of large solutions of second-order degenerate elliptic semilinear problems in non divergence form. The main particularity of the problem is the interior uniform ellipticity of the equation which degenerates on the boundary, involving an effect on the boundary blow up profile of the solution.

1. INTRODUCTION.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^2 boundary. This note is concerned with the problem

$$-\mathrm{Tr}(a(x)D^2u(x)) + u(x)^p = 0, \quad x \in \Omega, \tag{1.1}$$

where $p > 1$ and $a : \bar{\Omega} \to \mathbb{R}^{N\times N}$ is continuous. The main feature of the problem is the non-uniform ellipticity of the second-order operator in $\bar{\Omega}$. The precise assumptions on the problem will be made clear later.

We are interested is the study of *large solutions* associated to (1.1), namely, solutions to this equation (say, classical) satisfying the *blow-up boundary condition*

$$u(x) \to +\infty, \quad x \to \partial\Omega,\ x \in \Omega. \tag{1.2}$$

At the late fifties, Keller [14] and independently Osserman [21] addressed the question of large solutions for elliptic semilinear equations with the form

$$-\Delta u + f(u) = 0 \quad \text{in } \Omega, \tag{1.3}$$

obtaining fundamental information for this problem in the case the nonlinearity f is a positive function satisfying the so-called *Keller-Osserman condition*

$$\int_1^{+\infty} \frac{ds}{\sqrt{F(s)}} < +\infty, \qquad \text{where} \quad F(s) = \int_0^s f(t)dt,$$

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extending the results of Bieberbach [2] for large solutions for equations associated to exponential nonlinearities. In particular, when $f(t) = t^p$, the above condition is verified when $p > 1$ and for this reason, in what follows we mainly refer to this power-type nonlinearity.

After these seminal works of Keller and Osserman, the study of large solutions for semilinear elliptic problems has been widely developed in different frameworks until these days. Concerning (1.3), existence, uniqueness and asymptotic behavior near the boundary can be found in [18, 1, 19, 15] among many others. The effect of the second-order diffusive operator (the Laplacian) and the reactive term (the nonlinearity) over functions defined through negative powers of the distance function allow to construct blow-up barriers, and a subsequent systematical use of the comparison principle leads to the well-posedness of the problem. The asymptotic behavior at the boundary of the solution is a byproduct of the built barriers, which in the case of (1.3) explode at the rate $\mathrm{dist}(x, \partial\Omega)^{-2/(p-1)}$. The mentioned strategy to address the problem is typically quoted as the *method of sub and supersolutions*.

Further interesting results such as radiality of large solutions [22] or domain influence of the blow up [7] are also available. Besides, it is worth to mention the deep connection of the study of large solutions with stochastic superprocesses like the so-called *Brownian snake*, see [8, 17] and references therein.

This work is greatly motivated by the study of large solutions for equations like (1.1) in the *uniformly elliptic* setting (that is, when a is definite positive on $\bar{\Omega}$) by Veron [25]. The author proves existence and uniqueness, together with the Laplacian-type blow-up profile $\mathrm{dist}(x, \partial\Omega)^{-2/(p-1)}$ near the boundary. This asymptotic behavior is recovered from the well-known behavior of $-\Delta u + u^p = 0$ in a half-space after a nice scaling argument, feasible by the uniform ellipticity of the operator.

A second source of motivation comes from the study of elliptic problems with x-dependent nonlinearity, see [20, 4, 5, 11, 16] and references therein. An interesting model problem is the equation

$$-\Delta u(x) + b(x)u(x)^p = 0, \quad x \in \Omega. \tag{1.4}$$

where $b \in C(\Omega)$ is a positive function which explodes at the boundary. This type of problems can be regarded as a particular case of our problem (1.1) in the sense of the degenerate ellipticity of the second-order term. For instance, problem (1.4) can be re-written (in the context of a typical model) as

$$-d(x)^\mu \Delta u(x) + u(x)^p = 0 \quad x \in \Omega,$$

where $\mu > 0$ and d is a function that agrees $\mathrm{dist}(x, \partial\Omega)$ near the boundary and is uniformly positive inside Ω. In [5] it is proven that the existence of large solutions necessarily requires that $\mu \in (0, 2)$ and the blow-up profile of the solution is $d(x)^{-(2-\mu)/(p-1)}$. We remark that the results for this type of problems mainly follow the method of sub and supersolutions with acute modifications of the arguments presented in the x-independent case, and this is possible basically because of the subtle interaction among the x-dependency of the problem and the second-order term.

By the above discussion, it is natural to address the elemental questions on large solutions to elliptic problems in which the second-order term shows degenerate ellipticity at one hand, and a more delicate dependency on the state variable on the other. Besides the fundamental result about the well-posedness of the problem, we are also interested in explore how the degeneracy of the operator affects the power profile of the solution.

The basic conditions over (1.1) are listed next.

(**A0**) *The matrix a is uniformly elliptic in each compact set $\Omega' \subset\subset \Omega$, and it can be decomposed as $a = \sigma\sigma^T$ on $\bar{\Omega}$, where $\sigma : \bar{\Omega} \to \mathbb{R}^{N\times N}$ is continuous.*

(**A1**) *Normal degeneracy at the boundary: For each $x \in \partial\Omega$*

$$\sigma(x)^T Dd(x) = 0.$$

In the discussion of the above assumptions, we start remarking that we follow the method of sub and supersolutions to get our results and therefore comparison principle is the key tool. In this direction, the interior uniform ellipticity and the matrix decomposition property in (A0) have close relation with the version of the comparison principle we use, and also with the type of result we have in mind. The matrix decomposition is classical in the context of viscosity solutions, particularly in what respects to the viscosity comparison principle, see [6]. Being applicable to very degenerate elliptic problems, this version of comparison principle has certain restrictions on the continuity of a (it must be at least 1/2-Hölder, see [13]). Of course this implies a restriction on σ, the square root of a, and as a consequence it also restricts the "speed" in which the operator degenerates on the boundary, degeneracy imposed by (A1). Note that this is actually a "normal" degeneracy because $Dd(x)$ agrees the inward unit normal to the booundary at $x \in \partial\Omega$. Thus, the interior uniform ellipticity in (A0) allows us to weaken the regularity requirements on a since the application of elliptic regularity results to locally bounded solutions of (1.1) permits to compare them *classically* with appropriate sub and supersolutions (see Proposition 2.2 below).

The main result of this paper is the following

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^2 boundary, $p > 1$ and assume (A0),(A1) hold. Then, there exists a unique solution $\bar{u} \in C^2(\Omega)$ to problem* (1.1)-(1.2) *in each of the following two cases:*

(i) Fast Degeneracy in Strictly Convex Sets: *The set Ω is strictly convex and the following two assumptions hold:*

Fast Normal Degeneracy*: There exist $L > 0$, and $1/2 \leq \theta \leq 1$ such that, for each $x \in \Omega$ near the boundary*

$$|\sigma^T(x)Dd(x)| \leq Ld^\theta(x). \tag{1.5}$$

Tangential Nondegeneracy*: There exists $\underline{\lambda} > 0$ such that for each $x \in \partial\Omega$ there exists $j \in \{1, , N\}$*

$$|\sigma_j(x)| \geq \underline{\lambda}, \tag{1.6}$$

where $\sigma_j(x)$ denotes the j-th column of $\sigma(x)$.

(ii) Controlled Slow Degeneracy in General Domains: *There exist $\bar{L} \geq L > 0$ and $0 < \theta < 1/2$ such that, for each $x \in \Omega$ near the boundary*

$$Ld(x)^\theta \leq |\sigma(x)^T Dd(x)| \leq \bar{L}d(x)^\theta. \tag{1.7}$$

Moreover, in both cases $(i), (ii)$, there exists $0 < c_1 \leq C_1$ such that

$$c_1 d^{-\gamma}(x) \leq \bar{u}(x) \leq C_1 d^{-\gamma}(x),$$

for all x near the boundary, where $\gamma > 0$ is defined as

$$\gamma = \begin{cases} 1/(p-1) & \text{in case (i)} \\ (2-2\theta)/(p-1) & \text{in case (ii).} \end{cases} \tag{1.8}$$

Concerning the case (i) in the above theorem, the convexity of Ω and the tangential nondegeneracy condition (1.6) seem to be a “boundary hitting” condition for the underlying stochastic process

$$dX_t = \sqrt{2}\sigma(X_t)^T dW_t, \ t > 0; \quad X_0 = x,$$

where W_t is the standard Brownian motion in $\mathbb{R}^N$ and for which our second-order operator is the infinitesimal generator (see [24]). Roughly speaking, the process (X_t) just moves tangentially to $\partial\Omega$ near the boundary by (A1), and the convexity and (1.6) allow the process to hit the boundary in finite time. In terms of the computations, the fast decay assumption (1.5) determines a leading effect of the curvature of the boundary on the second-order operator when we evaluate it on distance-type barriers near the boundary. This can be translated as a “first-order” effect of the operator and a balance of powers between the differential operator and the nonlinearity drives us to the above blow-up profile. Finally, notice that (1.5) holds for σ Hölder continuous with Hölder exponent θ.

On the other hand, the slow decay assumption (1.7) in case (ii) leads to an actual “second-order” effect of the differential operator and the accuracy of its decay allows us to get the result, which is consistent with the results obtained in [5]. We would like to remark that using the same arguments presented below, it is possible to consider $\theta = 1/2$ in case (ii), provided L in (1.7) is large enough in terms of the curvature of $\partial\Omega$ and the L^∞ bounds of a.

We finish mentioning that the continuity of the square root of a nonnegative matrix-valued function is studied in [3], see also [9, 10]. In terms of our notation, the authors prove that a can always be decomposed as in (A0) and its square root is continuous too. Using this result and the arguments showed below, if we assume that

$$|\sigma^T Dd| > 0 \quad \text{on } \partial\Omega \tag{1.9}$$

we are able to get existence, uniqueness and Laplacian-type asymptotic behavior to (1.1), extending the results given in [25]. Moreover, we can get the same result always assuming (1.9) but allowing some boundary degeneracy of the second-order operator in "non-normal" directions.

Basic Notation: For a set $\mathcal{O} \subset \mathbb{R}^N$ we denote $d_{\mathcal{O}}$ the signed distance function to $\partial\mathcal{O}$ which is positive in $\mathcal{O}$ and we simply write d in the case $\mathcal{O} = \Omega$. For $\delta > 0$ we denote

$$\Omega_\delta = \{x \in \Omega : d(x) < \delta\}.$$

By the smoothness assumption over Ω, there exists $\delta_0 > 0$ such that $d \in C^2(\Omega_{\delta_0})$, see [12].

2. Existence

We start with some direct computations concerning powers of the distance function. For $\gamma \neq 0$ and $\epsilon \geq 0$, we see that

$$D(d+\epsilon)^\gamma(x) = \gamma(d(x)+\epsilon)^{\gamma-1} Dd(x),$$
$$D^2(d+\epsilon)^\gamma(x) = \gamma(d(x)+\epsilon)^{\gamma-1}\Big((\gamma-1)(d(x)+\epsilon)^{-1} Dd(x)\otimes Dd(x) + D^2 d(x)\Big),$$

for each $x \in \Omega_{\delta_0/2}$. Therefore, recalling $a = \sigma\sigma^T$, for all $x \in \Omega_{\delta_0/2}$ and $\epsilon \geq 0$ small, we conclude that

$$\begin{aligned} &\mathrm{Tr}\Big(a(x) D^2(d+\epsilon)^\gamma(x)\Big) \\ &= \gamma(d(x)+\epsilon)^{\gamma-1}\Big((\gamma-1)(d(x)+\epsilon)^{-1}|\sigma^T(x)Dd(x)|^2 + \mathrm{Tr}(a(x)D^2 d(x))\Big). \end{aligned} \tag{2.1}$$

Now we provide some useful estimates concerning the terms arising in (2.1). We start mentioning some facts related with the smoothness and strict convexity assumptions over $\partial\Omega$, this last assumption playing a key role in the case (i) of Theorem 1.1.

Notice that for each $x \in \Omega_{\delta_0}$, we can decompose $D^2 d(x)$ as

$$D^2 d(x) = P\mathcal{D}P^T,$$

where $P = P(x)$ is a matrix whose column vectors form an orthonormal basis of $\mathbb{R}^N$ (and whose N-th column equals $Dd(x)$) and

$$\mathcal{D} = \mathcal{D}(x) = \mathrm{diag}[-\kappa_1/(1-\kappa_1 d), ..., -\kappa_{N-1}/(1-\kappa_{N-1} d), 0],$$

with $\kappa_j = \kappa_j(x), j = 1, ..., N-1$ are the principal curvatures of $\partial\Omega$ at the point x (see [12]). By the smoothness of $\partial\Omega$ there exists $\bar\kappa > 0$ such that

$$k_j(x) \leq \bar\kappa \quad \text{for all } j = 1, ...N-1, \ x \in \Omega_{\delta_0}, \tag{2.2}$$

meanwhile, when Ω is strictly convex, there exists $\underline{\kappa} > 0$ such that

$$\underline{\kappa} \leq k_j(x) \quad \text{for all } j = 1, ...N-1, \ x \in \Omega_{\delta_0}. \tag{2.3}$$

The role of the strict convexity assumption over the boundary is given in the following

Lemma 2.1. *Assume Ω is strictly convex and that a satisfies (A0) and* (1.6)*. Then, there exists a constant $\underline{c} > 0$ and $\bar\delta > 0$ such that*

$$\mathrm{Tr}(a(x)D^2 d(x)) \leq -\underline{c}, \quad \textit{for all } x \in \Omega_{\bar\delta}.$$

Proof: We start assuming $\bar\delta < \delta_0$. By (A0) we can write

$$\mathrm{Tr}(a(x)D^2 d(x)) = \sum_{j=1}^N \langle D^2 d(x)\sigma_j(x), \sigma_j(x)\rangle,$$

where $\sigma_j(x) \in \mathbb{R}^N$ denotes the j-th row of $\sigma(x)$. Then, using the decomposition of Dd we arrive at

$$\mathrm{Tr}(a(x)D^2 d(x)) = \sum_{j=1}^N \langle \mathcal{D}(x)P^T(x)\sigma_j(x), P^T(x)\sigma_j(x)\rangle$$

and then, considering $d(x) \leq \bar\kappa/2$ we can write

$$\mathrm{Tr}(a(x)D^2 d(x)) \leq -2\underline{\kappa}\sum_{j=1}^N |P^T(x)\sigma_j(x)|^2,$$

and by (1.6) we conclude that $\mathrm{Tr}(a(x)D^2 d(x)) \leq -2\underline{\kappa}\bar\lambda$. □

Proposition 2.2. *Assume hypotheses of case* (*i*) *or* (*ii*) *in Theorem 1.1 hold. Then, for each $R > 1$ there exists a unique solution $u_R \in C^2(\Omega)\cap C(\bar\Omega)$ to problem* (1.1) *satisfying the boundary condition $u = R$ on $\partial\Omega$.*

Proof: For each $h > 0$ small, consider $\Omega^h = \Omega \setminus \Omega_h$, which is bounded and smooth. Then, by (A0) the matrix a is uniformly elliptic in Ω^h and using standard elliptic techniques we have the existence of a unique function $u^h \in C^2(\Omega^h) \cap C(\bar\Omega^h)$ solving the problem $-\mathrm{Tr}(aD^2u) + u^p = 0$ in Ω^h, and satisfying the boundary condition $u^h = R$ on $\partial\Omega^h$, see Lemma 3.3 of [25]. By comparison principle we see that $0 \leq u^h \leq R$ on $\bar\Omega^h$, for each $h > 0$ small.

Now, denote $d_h = d_{\Omega^h}$ which is smooth in $(\Omega^h)_{\delta_0/2}$, uniformly in h small enough. Consider $C > 0$, $\tilde\gamma \in (0,1)$ to be fixed and denote $\rho = (R/C)^{1/\tilde\gamma}$. Take $C > 0$ large enough in terms of R and δ_0 to have $\rho < \delta_0/4$. Hence, the function $\psi = R - Cd_h^{\tilde\gamma}$ is smooth in $(\Omega^h)_\rho$.

Notice that by the choice of ρ, $\psi \le u^h$ on $\partial\Omega^h$ and in what follows we prove that ψ is a subsolution to (1.1) in $(\Omega^h)_\rho$. A similar computation to (2.1) together with (A0) lead us to

(2.4)
$$
\begin{aligned}
&-\operatorname{Tr}(a(x)D^2\psi(x))\\
&= C\tilde\gamma(\tilde\gamma-1)d_h(x)^{\tilde\gamma-2}|\sigma^T(x)Dd_h(x)|^2 + C\tilde\gamma d_h(x)^{\tilde\gamma-1}\operatorname{Tr}(a(x)D^2d_h(x)).
\end{aligned}
$$

At this point we split the analysis. In case (i) it is easy to note that if Ω is strictly convex, then for each h small Ω^h is strictly convex too. Then, by Lemma 2.1 we get that

$$-\operatorname{Tr}(a(x)D^2\psi(x)) \le -C\ c\ d_h(x)^{\tilde\gamma-1}$$

for all $x \in (\Omega^h)_\rho$, for some $c > 0$ not depending on h. Thus, we can write

$$-\operatorname{Tr}(aD^2\psi) + \psi^p \le -C\ c\ d_h^{\tilde\gamma-1} + R^P \le 0 \quad \text{in } (\Omega^h)_\rho,$$

by taking C large in terms of R and the data.

In case (ii), we start noting that for all $x \in \Omega_{\delta_0}$ we have $Dd(x) = Dd_h(x)$. In fact, denoting $\hat x$ its unique projection to $\partial\Omega$ and x_h its unique projection to $\partial\Omega^h$, the above claim is equivalent to prove that $x, \hat x, x_h$ belong to the same line. Thus, denoting x^* the unique point in the intersection of $\partial\Omega^h$ and the straight line joining x and $\hat x$, by contradiction we assume $x^* \neq x_h$. Notice that the projection of x^* to $\partial\Omega$ must be $\hat x$, and denoting $\hat x_h$ the unique projection of x_h to $\partial\Omega$, we see that $x^* \neq x_h$ implies that $\hat x_h \neq \hat x$. Thus, we can write

$$d(x) = |x - x^*| + |x^* - \hat x| > d_h(x) + h \ge |x - x_h| + |x_h - \hat x_h| \ge |x - \hat x_h|,$$

which contradicts the definition of $d(x)$.

Hence, we use this and the assumptions in case (ii) to get

$$|\sigma(x)^T Dd_h(x)| = |\sigma(x)^T Dd(x)| \ge Cd(x)^\theta \ge Cd_h(x)^\theta,$$

and plugging this into (2.4) we have the existence of a constants $c_1, c_2 > 0$ not depending on h or R such that

$$-\operatorname{Tr}(a(x)D^2\psi(x)) \le -Cd_h(x)^{\tilde\gamma+2\theta-2}(c_1 - c_2 d_h(x)^{1-2\theta}).$$

Then, taking $\rho \le (c_1/(2c_2))^{1/(1-2\theta)}$ (this means C large in terms of the data and R, but not on h), we get

$$-\operatorname{Tr}(aD^2\psi) + \psi^p \le -Cc_1 d_h(x)^{\tilde\gamma+2\theta-2}/2 + R^p \le 0 \quad \text{in } (\Omega^h)_\rho,$$

where the last inequality holds by enlarging C in terms of R and the data if it is necessary.

Therefore, in both cases (i) and (ii), using comparison principle we get that the solution u_h satisfies

$$(R - Cd_h(x)^{\tilde\gamma})_+ \le u^h(x) \le R, \quad x \in \Omega^h,$$

where the constant $C > 0$ depends only on R and the data, but not on h. Note the family $\{u^h\}$ is uniformly bounded in terms of h, by comparison it is monotone decreasing in compact sets of Ω and since equation (1.1) is locally uniformly elliptic in Ω, elliptic estimates provide us the compactness to conclude the existence of $u_R \in C^2(\Omega)$ such that $u^h \to u_R$ in $C^2_{loc}(\Omega)$. Of course $u_R \leq R$ in Ω. Now, for each $x \in \Omega$ close to the boundary, for each $h \leq d(x)/2$ we can write

$$u_R(x) \geq u^h(x) - |u_R(x) - u^h(x)| \geq R - Cd_h(x)^{\tilde{\gamma}} - o_h(1),$$

and by taking $h \to 0$, we conclude $u_R(x) \geq R - Cd(x)^{\tilde{\gamma}}$. The proof is finished. □

The following lemmas are the crucial in the existence proof.

Lemma 2.3. *Let $d \in C^2(\bar{\Omega})$ be a strictly positive function in $\Omega \setminus \Omega_{\delta_0}$ that coincides with $d(\cdot, \partial\Omega)$ in Ω_{δ_0}. Let γ as in* (1.8) *and for $K_1, K_2 > 0$ consider the function*

$$\Psi_+(x) = K_1 d^{-\gamma}(x) + K_2$$

Then, there exists K_1, K_2 such that Ψ_+ is a supersolution to (1.1) *in Ω.*

Proof: Recall $\bar{\delta} \in (0, \delta_0)$ in Lemma 2.1. For $x \in \Omega \setminus \Omega_{\bar{\delta}}$ both cases can be proven at the same time: by the strict positivity and smoothness of d in $\Omega \setminus \Omega_{\bar{\delta}}$, there exists a constant $C = C(\bar{\delta}) > 0$ such that

$$-\mathrm{Tr}(a(x)D^2\Psi_+(x)) + \Psi_+^p(x) \geq -K_1 C + K_2^p$$

and then we conclude that Ψ_+ is a supersolution to (1.1) in $\Omega \setminus \Omega_{\bar{\delta}}$ provided K_2 is large in terms of C and K_1. By the continuity of $D^2 d$ and a on $\bar{\Omega}$ we see that $\mathrm{Tr}(aD^2 d)$ is bounded below on $\bar{\Omega}$.

For $x \in \Omega_{\bar{\delta}}$, we split the analysis. In the case (i), using (2.1) and the fast decay assumption, Lemma 2.1 together with the definition of γ in this case lead to the existence of $C, c > 0$ just depending on the data such that

$$\begin{aligned} & -\mathrm{Tr}(a(x)D^2\Psi_+(x)) + \Psi_+^p(x) \\ \geq & -K_1 C\Big(d(x)^{-\gamma-2+2\theta} + cd(x)^{-(\gamma+1)}\Big) + K_1^p d(x)^{-(\gamma+1)}, \end{aligned}$$

and therefore, since $\theta \geq 1/2$ we can write

$$-\mathrm{Tr}(a(x)D^2\Psi_+(x)) + \Psi_+^p(x) \geq K_1 d^{-(\gamma+1)}(x)(-C + K_1^{p-1}).$$

Thus, taking K_1 large just in terms of the data, we conclude Ψ_+ is a supersolution to (1.1).

For case (ii) we use (1.7) to find that there exists a constants $C, c > 0$ such that

$$\begin{aligned} & -\mathrm{Tr}(a(x)D^2\Psi_+(x)) + \Psi_+^p(x) \\ \geq & -K_1 C\Big(d(x)^{-\gamma-2+2\theta} - cd(x)^{-(\gamma+1)}\Big) + K_1^p d(x)^{-(\gamma+1)}, \end{aligned} \tag{2.5}$$

Now, using that $\theta < 1/2$ and the definition of γ in case (ii), we see that Ψ_+ is a supersolution to (1.1) taking K_1 large just in terms of the data. $\square$

Lemma 2.4. *Let $d \in C^2(\bar{\Omega})$ be a strictly positive function in $\Omega \setminus \Omega_{\delta_0}$ that coincides with $d(\cdot, \partial\Omega)$ in Ω_{δ_0}. Let γ as in (1.8) and for $K_1, K_2, \epsilon > 0$ consider the function*

$$\Psi_-(x) = K_1(d(x) + \epsilon)^{-\gamma} - K_2.$$

Then, there exist $K_1, K_2, \bar{\delta} > 0$ such that $\Psi_- \leq 0$ on $\Omega \cap \partial\Omega_{\bar{\delta}}$ and it is a subsolution to (1.1) in $\Omega_{\bar{\delta}}$, for each $\epsilon \in (0, \bar{\delta}/4)$.

Proof: We start considering $\bar{\delta}$ as in Lemma 2.1 (recall that $\bar{\delta} < \delta_0$) and consider $K_1, K_2 > 0$ satisfying the ratio $(K_1/K_2)^{1/\gamma} \leq \bar{\delta}$.

We start with the case (i). For $x \in \Omega_{\bar{\delta}}$, using (2.1) and the definition of γ we can write

$$\begin{aligned} & - \operatorname{Tr}(a(x)D^2\Psi_-(x)) + \Psi_-^p(x) \\ \leq & - K_1\gamma(\gamma+1)(d(x)+\epsilon)^{-(\gamma+2)}|\sigma^T(x)Dd(x)|^2 \\ & + \gamma(d(x)+\epsilon)^{-(\gamma+1)}\operatorname{Tr}(a(x)D^2d(x)) + K_1^p(d(x)+\epsilon)^{-(\gamma+1)}. \end{aligned} \tag{2.6}$$

Next, dropping the first term in the right-hand side of the above inequality and applying Lemma 2.1 we have the existence of a constant $c > 0$ just depending on the data such that

$$-\operatorname{Tr}(a(x)D^2\Psi_-(x)) + \Psi_-^p(x) \leq -K_1(d(x)+\epsilon)^{-(\gamma+1)}(c - K_1^{p-1})$$

from which we conclude the result by taking K_1 small in terms of c.

In case (ii), for $x \in \Omega_{\bar{\delta}}$ the same inequality (2.6) holds. This time we use the slow decay assumption (1.7) and the boundedness of a and D^2d on $\bar{\Omega}$ to conclude the existence of two constant $C, c > 0$ just depending on the data such that

$$\begin{aligned} & - \operatorname{Tr}(a(x)D^2\Psi_-(x)) + \Psi_-^p(x) \\ \leq & - K_1 c\Big((d(x)+\epsilon)^{-\gamma-2+2\theta} - C(d(x)+\epsilon)^{-(\gamma+1)}\Big) + K_1^p(d(x)+\epsilon)^{-(\gamma+1)}, \end{aligned}$$

but since in this case $\theta < 1/2$ and eventually taking $\bar{\delta}$ smaller, we arrive at

$$-\operatorname{Tr}(a(x)D^2\Psi_-(x)) + \Psi_-^p(x) \leq -K_1(d(x)+\epsilon)^{-(\gamma+1)}(c/2 - K_1^{p-1}),$$

from which we conclude the result taking K_1 small. $\square$

***Proof of Theorem 1.1* - Existence:** Let u_R be the unique solution given in Proposition 2.2. Using comparison principle we see that $0 \leq u_R \leq R$ and the sequence $\{u_R\}$ is nondecreasing in R. Moreover, properly using comparison principle again, we conclude $u_R \leq \Psi_+$ in Ω by Lemma 2.3 and therefore the family $\{u_R\}$ is unformly bounded in $L^\infty_{loc}(\Omega)$. Then, by the uniform ellipticity of the second-order operator on compact sets of Ω given by (A0) together with elliptic regularity estimates and its monotony, by a bootstrapping argument we conclude the compactness of $\{u_R\}$ in $C^2_{loc}(\Omega)$. This leads us to the existence of a classical solution $\underline{u} \in C^2$ to (1.1) such that $u_R \to \underline{u}$ in $C^2_{loc}(\Omega)$ as $R \to \infty$. Recalling the definition of Ψ_- in Lemma 2.4, setting $\epsilon_R = (R/K_1)^{-1/\gamma}$ and for each $\epsilon \leq \epsilon_R$, by comparison principle we get $\Psi_- \leq u_R \leq \underline{u}$ in $\Omega_{\bar{\delta}}$. Letting $\epsilon \to 0$ we get a lower bound of explosion to $\underline{u}$. In fact, the above analysis leads us to the existence of two constants $0 < c_1 \leq C_1$ such that

$$c_1 d(x)^{-\gamma} \leq \underline{u}(x) \leq C_1 d(x)^{-\gamma}, \tag{2.7}$$

where γ satisfies (1.8). By construction, $\underline{u}$ is the minimal large solution to (1.1). □

3. Uniqueness

The key ingredient in the uniqueness proof is the following

Lemma 3.1. *Let γ be as in* (1.8)*. There exists constants $0 < c_1 \leq C_1 < +\infty$ such that, for each u solution to* (1.1)-(1.2) *we have*

$$c_1 d(x)^{-\gamma} \leq u(x) \leq C_1 d(x)^{-\gamma}, \quad \text{for all } x \in \Omega.$$

Proof: Recalling that the solution $\bar{u}$ found in Theorem 1.1 is the minimal solution among all large solutions to (1.1), we can take c_1 as in (2.7). Now we deal with the upper bound.

Let $\bar{\delta} > 0$ as in Lemmas 2.1 and 2.4, consider $\delta \in (0, \bar{\delta}/4)$ and define

$$A_\delta = \{x \in \Omega : \delta < d(x) < 3\delta\}.$$

Denote $\mu = 2/(p-1)$ and consider the function

$$\Psi(x) = \Psi_\delta(x) = k_\delta \Big((3\delta - d(x))^{-\mu} + (d(x) - \delta)^{-\mu} \Big),$$

where $k_\delta = k\delta^\eta$, with $\eta = 1/(p-1)$ in case (i), $\eta = 2\theta/(p-1)$ in case (ii), and $k > 0$ a constant to be fixed later. Let $x \in A_\delta$ and note that a direct computation leads us to

$$\begin{aligned} D^2\Psi(x) =& \mu k_\delta \Big[(\mu+1)\Big((d(x)-\delta)^{-(\mu+2)} + (3\delta - d(x))^{-(\mu+2)} \Big) Dd(x) \otimes Dd(x) \\ &+ \Big((3\delta - d(x))^{-(\mu+1)} + (d(x) - \delta) \Big) D^2 d(x) \Big]. \end{aligned}$$

Using (A0), together with Lemma 2.1 in case (i), a similar computation to (2.1) drives us to the existence of a constant $C > 0$ not depending on δ such that

$$\begin{aligned} -\mathrm{Tr}(a(x)D^2\Psi(x)) \geq & -Ck_\delta |\sigma^T(x)Dd(x)|^2 \Big((3\delta - d(x))^{-(\mu+2)} + (d(x)-\delta)^{-(\mu+2)} \Big) \\ & - Ck_\delta (d(x)-\delta)^{-(\mu+1)} - \chi C k_\delta (3\delta - d(x))^{-(\mu+1)}, \end{aligned}$$

where $\chi = 0$ in case (i) and $\chi = 1$ in case (ii). From here, using the fast/slow decay assumptions (c.f. (1.5)/(1.7)) for each case, we obtain

(3.1)
$$\begin{aligned} -\mathrm{Tr}(a(x)D^2\Psi(x)) \geq & -Ck_\delta d(x)^{2\theta}\Big((3\delta - d(x))^{-(\mu+2)} + (d(x)-\delta)^{-(\mu+2)}\Big) \\ & - Ck_\delta (d(x)-\delta)^{-(\mu+1)} - \chi C k_\delta (3\delta - d(x))^{-(\mu+1)}, \end{aligned}$$

for some constant $C > 0$ not depending on δ, and where $\theta \geq 1/2$ in case (i) and $\theta < 1/2$ in case (ii).

On the other hand, observing that there exists $c_p > 0$ such that

$$(\alpha+\beta)^p \geq c_p(\alpha^p + \beta^p) \quad \text{for all } \alpha, \beta > 0,$$

by definition of γ we see that

$$\Psi^p(x) \geq c_p k_\delta^p \Big((3\delta - d(x))^{-(\mu+2)} + (d(x)-\delta)^{-(\mu+2)}\Big).$$

Hence, using this and (3.1) we arrive at

$$\begin{aligned} & -\mathrm{Tr}(a(x)D^2\Psi(x)) + \Psi(x)^p \\ \geq & k(3\delta - d(x))^{-(\mu+2)}\Big(c_p k^{p-1}\delta^{p\eta} - C\delta^\eta d(x)^{2\theta} - \chi C\delta^\eta (3\delta - d(x))\Big) \\ & + k(d(x)-\delta)^{-(\mu+2)}\Big(c_p k^{p-1}\delta^{p\eta} - C\delta^\eta d(x)^{2\theta} - C\delta^\eta (d(x)-\delta)\Big). \end{aligned}$$

Thus, recalling that $d(x) \in (\delta, 3\delta)$ and the definition of η for each case, taking k large in terms of c_p and C in the above inequality, we conclude that Ψ is a blow-up supersolution to (1.1) in A_δ. Then, by comparison principle, each solution $u \in L^\infty_{loc}(\Omega)$ to (1.1) satisfies

$$u(z) \leq \Psi_\delta(z) \quad \text{for all } z \in A_\delta,$$

for each $\delta \in (0, \bar{\delta}/4)$.

Now, consider $x \in \Omega$ and denote $\delta = d(x)/2$. Then, using the last inequality, for each large solution $u \in L^\infty_{loc}(\Omega)$ to (1.1) we get that

$$u(x) \leq 2^{1+\mu} k d(x)^{-\gamma},$$

which concludes the proof. □

We need to introduce the following definition: let $\{\Omega_m\}_m$ be an exhausting sequence of smooth subdomains of Ω, that is, $\Omega_m \subset\subset \Omega_{m+1} \subset\subset \Omega$ for each $m \in \mathbb{N}$, and $\bigcup_{m\in\mathbb{N}} \Omega_m = \Omega$. Let u_m be the minimal large solution to the problem

$$-\mathrm{Tr}(aD^2u) + u^p = 0 \quad \text{in } \Omega_m.$$

Then, $\bar{u} = \lim_{m\to\infty} u_m$ is called the *maximal solution* to (1.1). By elliptic regularity, the limit is taken in the $C^2_{loc}(\Omega)$ sense and therefore $\bar{u} \in C^2(\Omega)$. Note that $\bar{u}$ is actually a large solution and therefore the estimates given in Lemma 3.1 applies to $\bar{u}$. Moreover, by construction of $\bar{u}$ and comparison principle, if $v \in C^2(\Omega)$ is any large solution to (1.1) we have $v \leq \bar{u}$ in Ω.

The following lemma is a simple adaptation of Theorem 0.2 in [19] to our setting

Lemma 3.2. *Assume hypotheses of Theorem 1.1 hold. Let $\bar{u}$ be the maximal solution to* (1.1) *in Ω and assume the existence of constants $K > 1, \delta > 0$ just depending on the data such that*

$$0 \le \bar{u} \le Ku \quad \text{in } \Omega_\delta, \tag{3.2}$$

for each large solution u to (1.1)*. Then, there exists at most one large solution to* (1.1)*.*

Before giving the proof of the above lemma, next we provide its main corollary which is the

Proof of Theorem 1.1 - Uniqueness: Let u be a large solution to (1.1) and let $\bar{u}$ be the maximal solution to (1.1). Since $\bar{u}$ is also a large solution to the equation, the estimates in Lemma 3.1 apply to u and $\bar{u}$, from which we conclude that

$$\bar{u}(x) \le C_1 c_1^{-1} u(x) \quad x \in \Omega_\delta,$$

for some $\delta > 0$ just depending on the data. Defining $K = C_1 c_1^{-1}$ we can get the uniqueness from Lemma 3.2. □

We finally provide the

Proof of Lemma 3.2: Let u be a large solution to (1.1) and assume that $u \neq \bar{u}$. Denote $\mathcal{A} = \{x \in \Omega : \bar{u}(x) > u(x)\} \neq \emptyset$ and define

$$w = u - \frac{\bar{u} - u}{2K}.$$

The maximality of $\bar{u}$ implies $w \le u$ in Ω, and in addition we see that

$$w = \frac{(2K+1)u - \bar{u}}{2K} \ge \frac{(K+1)u}{2K} \quad \text{in } \Omega_\delta. \tag{3.3}$$

Note that for each $\lambda_1 > 1$, $f : \mathbb{R} \to \mathbb{R}$ convex and $a, b > 0$, denoting $\lambda_2 = \lambda_1 - 1$ we see that

$$f(a) = f(\lambda_1^{-1}(\lambda_1 a - \lambda_2 b) + \lambda_1^{-1}\lambda_2 b) \le \lambda_1^{-1} f(\lambda_1 a - \lambda_2 b) + \lambda_1^{-1}\lambda_2 f(b).$$

Thus, we use this inequality with $f(t) = t^p, p > 1$, $\lambda_1 = \frac{2K+1}{2K}$ and $a = u(x), b = \bar{u}(x)$ for $x \in \Omega$, to get

$$-\mathrm{Tr}(a(x)D^2 w(x)) + w^p(x) \ge 0 \quad \text{for } x \in \Omega,$$

and conclude that w is a supersolution to (1.1).

On the other hand, consider $0 < \theta < \frac{K+1}{2K} < 1$ and define $w_\theta = \theta u$. Then

$$-\mathrm{Tr}(aD^2 w_\theta) + w_\theta^p = -\theta \mathrm{Tr}(aD^2 u) + \theta^p u^p \le (\theta^p - \theta)u^p \le 0,$$

which means that w_θ is a subsolution to (1.1). By (3.3) and the choice of θ, we clearly see that $w_\theta < w$ in an open neighborhood of $\partial\Omega$ and then, by comparison principle, we conclude that $w_\theta \leq w$ in Ω. Thus, using a similar procedure as in the proof of existence of Theorem 1.1, it is possible to construct a solution $u_1 \in C^2(\Omega)$ to problem (1.1) such that $w_\theta \leq u_1 \leq w$. In particular, u_1 is a large solution and therefore it satisfies (3.2).

Note that the inequality $u_1 \leq w$ in Ω implies that

$$\bar{u} - u_1 \geq (1 + \frac{1}{2K})(\bar{u} - u) \quad \text{in } \Omega. \tag{3.4}$$

Performing the same analysis as above but replacing u by u_1 we can get a large solution u_2 to equation (1.1), satisfying (3.2) and, as in (3.4) it also satisfies

$$\bar{u} - u_2 \geq (1 + \frac{1}{2K})(\bar{u} - u_1) \quad \text{in } \Omega.$$

By an inductive argument, we can construct a sequence of large solutions u_n to (1.1) such that

$$\bar{u} - u_n \geq (1 + \frac{1}{2K})(\bar{u} - u_{n-1}) \quad \text{in } \Omega,$$

and therefore

$$\bar{u} \geq u_n + (1 + \frac{1}{2K})^n(\bar{u} - u) \quad \text{in } \Omega.$$

Since $u_n \geq 0$ and there exists at least one point $x \in \Omega$ such that $(\bar{u} - u)(x) > 0$, we arrive at a contradiction making $n \to \infty$. □

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