Math. Ann.
DOI 10.1007/s00208-016-1481-3

Mathematische Annalen

CrossMark

Continuous viscosity solutions for nonlocal Dirichlet problems with coercive gradient terms

Gonzalo Dávila[1] · Alexander Quaas[1] · Erwin Topp[2]

Received: 15 March 2016 / Revised: 25 August 2016

Abstract In this paper we study existence of solutions of nonlocal Dirichlet problems that include a coercive gradient term, whose scaling strictly dominates the one of the integro-differential operator. For such problems the stronger effect of the gradient term may give rise to solutions not attaining the boundary data or discontinuous solutions on the boundary. Our main result states that under suitable conditions over the right-hand side and boundary data, there is a (unique) Hölder continuous viscosity solution attaining the boundary data in the classical sense. This result is accomplished by the construction of suitable barriers which, as a byproduct, lead to regularity results up to the boundary for the solution.

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded, smooth domain, and fix $s \in (0, 1)$ and $p > 2s$. In this article we are interested in elliptic nonlocal problems of the form

$$\lambda u + (-\Delta)^s u + |Du|^p = f \quad \text{in } \Omega, \tag{1.1}$$

✉ Gonzalo Dávila
gonzalo.davila@usm.cl

Alexander Quaas
alexander.quaas@usm.cl

Erwin Topp
erwin.topp@usach.cl

1 Departamento de Matemática, Universidad Técnica Federico Santa María, Casilla: v-110, Avda. España 1680, Valparaíso, Chile

2 Departamento de Matemática y C.C., Universidad de Santiago de Chile, Casilla 307, Santiago, Chile

Published online: 05 October 2016

where $\lambda \geq 0$, $f \in C(\bar{\Omega})$ and $(-\Delta)^s$ denotes the fractional Laplacian of order $2s$, given by the formula

$$(-\Delta)^s u(x) = C_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(z)}{|x - z|^{N+2s}} dz, \tag{1.2}$$

whenever the integral term has a sense. Here P.V. stands for the Cauchy principal value and $C_{N,s} > 0$ is a well-known normalizing constant, see [18]. It is useful for latter purposes to write the equivalent expression

$$(-\Delta)^s u(x) = -\frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2s}}.$$

Equation (1.1) is coupled with the *exterior Dirichlet* condition

$$u = \varphi \quad \text{in } \Omega^c, \tag{1.3}$$

where $\varphi \colon \Omega^c \to \mathbb{R}$ is a bounded, continuous function.

We are interested in studying conditions over the data f and φ in order to get existence of a *continuous function* $u \colon \mathbb{R}^N \to \mathbb{R}$ solving (1.1) in the viscosity sense, satisfying (1.3) punctually (or classically). We stress on the fact that such a solution does not develop discontinuities on $\partial\Omega$. Since $p > 2s$ we cannot expect to find continuous solutions of our problem in the current general setting and therefore we must consider some conditions over the data.

In the local case $s = 1$ there are several results concerning both existence and regularity of solutions of

$$\begin{cases} \lambda u - \Delta u + |Du|^p = f \text{ in } \Omega \\ \qquad\qquad\qquad u = \varphi \text{ on } \partial\Omega, \end{cases} \tag{1.4}$$

where $p > 0$. Several important results regarding (1.4) are obtained by Lasry and Lions in [23]. In the particular *strictly proper* case ($\lambda > 0$), the authors proved (Theorem I.4, [23]) the existence and uniqueness of a classical (hence, viscosity) solution to (1.4) satisfying the boundary condition in the *generalized sense*.

Roughly speaking this generalized notion states that whenever the viscosity solution u does not attain the boundary condition, that is when $u(x_0) \neq \varphi(x_0)$, then it must satisfy the equation at $x_0 \in \partial\Omega$ (in the viscosity sense). See [16] for a robust introduction of this topic in the second-order setting.

In [4], Barles and Da Lio addressed the problem in the parabolic framework (which resembles the strictly proper case in (1.4)) and proved that in the *subquadratic* range $p \leq 2$ the generalized viscosity solution $u \in C(\bar{\Omega})$ satisfies $u = \varphi$ on $\partial\Omega$, that is, no loss of the boundary data occurs.

In the case $p > 2$, also called *supercritical*, the diffusion cannot longer control the gradient and therefore a phenomena of “elliptic degeneracy” occurs. As a consequence of this, solutions may not attain the boundary condition, see for example [29] and the references therein. A positive answer to this question is obtained by Capuzzo-Dolcetta, Leoni and Porretta in [14], but their result requires an extra subtle relation between λ, φ and f.

Another interesting topic is related to the boundary behavior of the gradient of solutions attaining (or not) the boundary data. For instance, in [1] it is proven that solutions have bounded gradient on $\partial\Omega$, meanwhile [25] and [4] provide examples with boundary gradient blow up. For this, the effect of the gradient term must be strong enough to break the regularizing effect of the Laplacian, and this subtle interaction in limit cases may lead to solutions not satisfying the boundary condition and/or having boundary gradient blow up.

It is worth to point out the deep connection between the problem (1.4) and the stochastic optimal exit time problem. From the latter point of view, the role of the superquadratic gradient and the non attainability of the boundary condition becomes more evident, see [22,23,28] and references therein.

We finish this brief review on the second-order case with a comment on the role of the proper term λ. When this term is strictly positive then a priori bounds for the solution in L^∞ in terms of $\lambda^{-1}||f||_\infty$ are available. This is used in [23] to study the behavior of (1.4) as $\lambda \to 0^+$ (giving rise to the so-called *ergodic problem*), and concluding that if $\lambda = 0$ in (1.4) the problem may not be solvable for every right-hand side $f \in C(\bar{\Omega})$.

Concerning the nonlocal problem (1.1)–(1.3), we point out that existence and uniqueness of viscosity solutions $u \in C(\mathbb{R}^N)$ in the case $p \leq 2s$ is obtained by Barles, Chasseigne and Imbert in [3]. Their proof relies heavily in the interaction between the non-integrability of the kernel defining the fractional Laplacian and the shape of its support. In the case $\varphi = 0$ we can formally split the operator as

$$(-\Delta)^s u(x) = (-\Delta)^s_c u(x) + \lambda(x)u(x), \tag{1.5}$$

where

$$\lambda(x) := C_{N,s} \int_{\Omega^c} \frac{dz}{|x - z|^{N+2s}} > 0,$$

Here $(-\Delta)^s_c$ denotes the *censored fractional Laplacian of order* $2s$ defined as

$$(-\Delta)^s_c u(x) := C_{N,s} \text{P.V.} \int_\Omega \frac{u(x) - u(z)}{|x - z|^{N+2s}} dz,$$

see [11] for a deeper insight of this type of operator and its connection with censored s-stable Lévy processes.

Notice that the censored fractional operator in (1.5) is an x-dependent degenerate elliptic nonlocal operator in the sense of [5], uniformly elliptic in compact sets of Ω in the sense of [13], and since $\lambda(x) > 0$ this problem has a strictly proper structure, making the viscosity theory the appropriate framework to address this problem. In [3], the result is based on the leading effect of the λ-term in (1.5) since $\lambda(x) \nearrow +\infty$ as $x \to \partial\Omega$. The power profile of the explosion controls the contribution of the gradient term when this is of order $p \leq 2s$ and does not rely on the elliptic contribution of the censored operator.

In the supercritical fractional setting $p > 2s$, existence and uniqueness for problem (1.1)–(1.3) in the viscosity sense with *generalized boundary conditions* is obtained in [8], see Definition 2.2 for the precise statement of this notion of solution. However, in analogy with the second-order case, solutions may not attain the boundary condition in the classical sense. See also [3] for an explicit example of this phenomena for a linear equation with leading gradient term in the case $s < 1/2$.

In view of the previous discussion it is natural to ask for continuous solvability of (1.1)–(1.3) in the supercritical setting $p > 2s$. By following the arguments from [14] we are able to provide a positive answer to this question in the nonlocal setting. As expected from the local case, we need to impose certain assumptions on f, φ, λ and λ_0 (to be defined). More precisely we have the following theorem.

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^2 boundary, $\lambda \geq 0$ and $f \in C(\bar{\Omega})$. Let $s \in (1/2, 1)$, $p \in (2s, s/(1-s))$ and denote $\beta^* = (p-2s)/(p-1)$. Assume $\varphi \in C^{\beta^*}(\Omega^c)$ with $D := \operatorname{supp}\{\varphi\}$ compact, and for $x \in \Omega$ define the quantity*

$$\lambda_0(x) = C_{N,s} \int_{D^c} \frac{dz}{|x-z|^{N+2s}}.$$

Assuming the condition

$$\inf_{\Omega^c}\{\varphi\} \leq \inf_{\Omega} \left\{(\lambda + \lambda_0)^{-1} f\right\}, \tag{1.6}$$

there exists $M_0 > 0$ depending on the data such that if φ satisfies

$$|\varphi(x) - \varphi(y)| \leq M|x-y|^{\beta^*} \quad \text{for } x, y \in \Omega^c, \tag{1.7}$$

for some $M \leq M_0$, problem (1.1)–(1.3) has a unique viscosity solution in $C^{\beta^*}(\mathbb{R}^N)$.

As it can be seen in [8], the generalized solution u to (1.1)–(1.3) always satisfies $u \leq \varphi$ in Ω^c. Hence, the basic idea behind the proof of Theorem 1.1 is the construction of a subsolution to the problem attaining the boundary condition continuously, which is possible under the assumptions on the data given before. These conditions can be seen as the nonlocal counterpart to the condition exhibited in [14]. In fact, our results recover the ones in [14] by letting $s \to 1^-$.

The main ingredient on the construction of the barriers leading to the result is a power of the distance function with some exponent $\beta \in (0, 1)$. Hence, a natural balance of powers in the equation when we evaluate such a function leads to the equality $\beta - 2s = (\beta - 1)p$ which explains the value of β^* in the theorem. On the other hand, since we restrict ourselves to the case $p > 2s$ the previous balance of powers give rise to the condition $s > 1/2$.

Note that our result still provides existence when $\lambda + \inf_\Omega\{\lambda_0\} > 0$ which is a main difference with the local case. This can be explained due to the presence of the "proper" term λ_0 in (1.6) coming from the nonlocal nature of the operator, which has a close relation to $\lambda(x)$ in the formal splitting (1.5). Notice though that we cannot rely on the censored Laplacian to prove the result, since its diffusion is too weak to guarantee existence of solutions attaining the boundary data. In fact it is known that weak diffusive operators like the censored Laplacian and zero-th order operators may present solutions not achieving the boundary condition even in the absence of gradient terms (see [11,15,21]).

We point out we can address the problem in a more general setting but have presented it first in the context of (1.1)–(1.3) for simplicity. In fact, we can consider a nonlinear nonlocal operator and φ not compactly supported. In Theorem 2.3 we prove an analogous result as Theorem 1.1 for a family of nonlinear operators that possess an equicontinuity property. In Theorem 5.1 we extend the previous result to a general class of nonlinear operators with an additional restriction on the upper bound of the allowed powers of p.

Once we solve the Dirichlet problem with classical boundary condition, standard arguments allows us to conclude that the solution is in $C^{\beta}(\bar{\Omega})$ with $\beta = (p-2s)/(p-1)$, extending previous a priori regularity results up to the boundary for similar problems found in [6]. Recall that $p < s/(1-s)$ and therefore $\beta < s$, which is to be expected since s harmonic functions are C^s near the boundary, see [27].

The paper is organized as follows: in Sect. 2 we provide the basic notation and the notion of solution we use throughout the paper. In Sect. 3 we developed the main technical lemmas needed for the proofs of Theorem 2.3 and Theorem 5.1. In Sect. 4 we solve the Dirichlet problem for supercritical gradient terms under additional assumptions over the data proving Theorem 2.3. The second part of Sect. 4 is dedicated to extend our results when the fractional Laplacian is replaced by a particular family of nonlinear operators. Finally in Sect. 5 we extend the results to a family of nonlinear operators controlled by the usual fractional Pucci operators and provide some remarks regarding the pass to the limit $s \to 1^-$.

2 Main assumptions and statement of the problem

2.1 Basic notation and definition of solution

We start this section by introducing the notation used throughout the paper.

Let $r > 0$, we denote by $B_r(x)$ the ball centered at $x \in \mathbb{R}^N$ or just B_r when $x = 0$. For a domain $\Omega \subset \mathbb{R}^N$ we denote $d = d_\Omega$ the (signed) distance function to its boundary which is positive in Ω and non positive in Ω^c. For $\delta > 0$ we write $\Omega_\delta = \{x \in \Omega \colon d(x) < \delta\}$. When the domain is C^2, there exists $\delta_0 > 0$ such that $d \in C^2(\Omega_{\delta_0})$, see [24].

We use the notion of viscosity solution with generalized boundary condition as presented in [3,30], which is the natural extension to nonlocal problems of the notion for second-order equations presented in [16].

Throughout this article, we fix constants $0 < \gamma \le \Gamma < +\infty$ and denote by $\mathcal{K}$ the family of kernels K satisfying the inequalities

$$\gamma\, C_{N,s} \le K(y) \le \Gamma\, C_{N,s} \quad \text{for all } y \in \mathbb{R}^N, \tag{2.1}$$

where $C_{N,s} > 0$ is the constant arising in the definition of the fractional Laplacian in (1.2).

We point out that the normalizing constant in $C_{N,s} > 0$ appearing in the definition of the fractional Laplacian is always considered, even if we do not write it explicitly.

For $K \in \mathcal{K}$ we define the linear operator

$$L_K(u,x) := \int_{\mathbb{R}^n} \delta(u,x;y)\frac{K(y)}{|y|^{N+2s}}\,dy, \qquad \text{where} \quad \delta(u,x;y) = u(x+y)+u(x-y)-2u(x), \tag{2.2}$$

which is well defined if for each function $u : \mathbb{R}^N \to \mathbb{R}$ satisfying adequate regularity assumptions on x and weighted integrability assumptions at infinity, say $u \in C^{1,1}$ in a neighborhood of x and $u \in L^1(1/(1+|x|)^{N+2s})$.

Remark 2.1 Note that K represents the density with respect to the Levy measure $1/|z|^{N+2s}$. As an abuse of notation we will refer to K as a kernel.

The nonlinear nonlocal operator we have in mind considers a two-parameter family of kernels $\{K_{ab}\}_{a\in A, b\in B} \subseteq \mathcal{K}$ and writing $L_{a,b} := L_{K_{a,b}}$ we define

$$\mathcal{I}(u,x) := \inf_{a\in A}\sup_{b\in B} L_{a,b}(u,x), \tag{2.3}$$

which is well-defined under the same assumption on u described above. Recall that these type operators are known as Isaacs operators and they appear in a natural way in stochastic differential games (see for example [12] and [10] and the references therein).

Now we introduce notation to precisely describe the notion of solution. For $K \in \mathcal{K}$ and $D \subset \mathbb{R}^N$ we denote the restricted evaluation by

$$L_K[D](u,x) = \int_D \delta(u,x,z)K(z)|z|^{-(N+2s)}dz. \tag{2.4}$$

For a nonlinear operator as in (2.3), this restricted evaluation reads as

$$\mathcal{I}[D](u,x) = \inf\nolimits_{a\in A}\sup\nolimits_{b\in B} L_{a,b}[D](u,x),$$

and concerning the whole equations, for $\delta > 0$ we write

$$E_\delta(u,\phi,x) := -\mathcal{I}[B_\delta](\phi,x) - \mathcal{I}[B_\delta^c](u,x) + |D\phi(x)|^p - f(x).$$

Definition 2.2 A function $u : \mathbb{R}^N \to \mathbb{R}$ bounded and upper semicontinuous (usc for short) in $\bar{\Omega}$ is a viscosity subsolution to the Dirichlet problem (1.1) at $x_0 \in \bar{\Omega}$ if $u \le \varphi$ in $(\bar{\Omega})^c$ and if for each $\delta > 0$ and $\phi \in C^2(\mathbb{R}^N)$ such that x_0 is a maximum point of $u^* - \phi$ in $B_\delta(x_0)$, then

$$\begin{aligned} &E_\delta(u^*,\phi,x_0) \le 0 && \text{if } x_0 \in \Omega,\\ &\min\{E_\delta(u^*,\phi,x_0), u(x_0)-\varphi(x_0)\} \le 0 && \text{if } x_0 \in \partial\Omega, \end{aligned}$$

where u^* denotes the usc envelope of the function u in $\mathbb{R}^N$. In the analogous way, we define supersolutions and solutions to (1.1).

Recall that the usc envelope for a bounded function u is defined as

$$u^*(x) = \limsup_{y \to x} u(x), \quad \text{for all} \quad x \in \mathbb{R}^N.$$

By definition is also the smallest upper semicontinuous function which is larger or equal to u.

The above definition is given for data φ in L^∞, but a straightforward extension $\varphi \in L^1(w)$ (with certain growth at infinity in terms of s) can be stated.

2.2 The main result

In Sects. 3 and 4 we will focus our attention to a subclass of $\mathcal{K}$ satisfying certain equicontinuity properties.

More precisely, we consider $\bar{\mathcal{K}} \subset \mathcal{K}$ a family of kernels K satisfying the following property: for each $R > 0$ there exists a modulus of continuity m_R such that, for all $K \in \bar{\mathcal{K}}$

$$|K(x) - K(y)| \leq m_R(|x-y|), \quad \text{for all } x, y \in B_R. \tag{2.5}$$

Notice that we always can assume that $m_r(t) \leq m_R(t)$ for each $0 < r \leq R$ and $t \in (0, r)$.

In order to state our result for nonlinear nonlocal operators like (1.1), we assume that we can write $\bar{\mathcal{K}} = \{K_{a,b}\}_{a \in A, b \in B}$ for some sets of indices A, B.

In order to state our main result we require further definitions: for $\delta > 0$ we consider $\mathcal{N}_\delta = \Omega + B_\delta$, and notice that $\Omega \subset \mathcal{N}_\delta$ for all $\delta > 0$ (we denote $\mathcal{N}_0 = \Omega$).

For $\delta \geq 0$ and $x \in \Omega$ define

$$\lambda_{0,+}^\delta(x) = \sup_{K \in \bar{\mathcal{K}}} \int_{\mathcal{N}_\delta^c - x} \frac{(K(z) + K(-z))dz}{|z|^{N+2s}}, \tag{2.6}$$

and

$$\lambda_{0,-}^\delta(x) = \inf_{K \in \bar{\mathcal{K}}} \int_{\mathcal{N}_\delta^c - x} \frac{(K(z) + K(-z))dz}{|z|^{N+2s}}. \tag{2.7}$$

For $x \in \bar{\Omega}$ and $\delta > 0$ we also denote

$$f_\varphi^\delta(x) = f(x) + \inf_{K \in \bar{\mathcal{K}}} \left(\int_{\mathcal{N}_\delta^c - x} \frac{\varphi(x+z)}{|z|^{N+2s}} (K(z) + K(-z))dz \right). \tag{2.8}$$

With the above definitions we are in shape to state our main result.

Theorem 2.3 *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^2 boundary, $\lambda \geq 0$, $\varphi \in C(\Omega^c)$ bounded and $f \in C(\bar{\Omega})$. For $s \in (1/2, 1)$ and $\bar{\mathcal{K}} = \{K_{a,b}\}_{a \in A, b \in B}$ satisfying (2.5), consider $\mathcal{I}$ defined in (2.3) associated to $\bar{\mathcal{K}}$.*

For $p \in (2s, s/(1-s))$ denote $\beta^* = (p-2s)/(p-1)$ and assume $\inf_{\mathcal{N}_\delta \setminus \Omega}\{\varphi\} \geq 0$ (*resp.* $\inf_{\mathcal{N}_\delta \setminus \Omega}\{\varphi\} < 0$) *and the existence of $\delta > 0$ such that*

$$\inf_{\mathcal{N}_\delta \setminus \Omega}\{\varphi\} \leq \inf_{\Omega}\left\{(\lambda + \lambda_{0,+}^{\delta})^{-1} f_{\varphi}^{\delta}\right\}, \tag{2.9}$$

(resp. replacing $\lambda_{0,+}^{\delta}$ by $\lambda_{0,-}^{\delta}$).

Then, there exists $M_\delta > 0$ depending on δ and the data such that if φ satisfies

$$|\varphi(x) - \varphi(y)| \leq M|x-y|^{\beta^*} \quad \text{for } x, y \in \mathcal{N}_\delta \setminus \Omega, \tag{2.10}$$

for some $M \leq M_\delta$, problem

$$\begin{cases} \lambda u - \mathcal{I}(u,x) + |Du|^p = f(x) & \text{in } \Omega \\ u = \varphi & \text{in } \Omega^c, \end{cases} \tag{2.11}$$

has a unique viscosity solution $u \in C(\mathbb{R}^N)$.

Moreover, the restriction of u to $\bar{\Omega}$ belongs to $C^{\beta^*}(\bar{\Omega})$.

Remark 2.4 The functions $\lambda_{0,\pm}^{\delta}(x)$ come from the contribution of the non-local operator in $\mathcal{N}_\delta^c$, that is the long jump of the process/long range interaction. These terms appear since φ is truncated by zero in $\mathcal{N}_\delta^c$ in order to reduce the problem to the compactly supported case. The change from f in the source term to f_φ^δ is also related to the truncation, for more details see Lemma 4.1.

Note that when $\gamma = \Gamma = 1/2$ we see that $\mathcal{I} = -(-\Delta)^s$ and therefore Theorem 1.1 is a direct consequence of the above result.

An example where condition (2.9) is immediately satisfied is the case $f \geq 0$ in $\bar{\Omega}$, $\varphi \leq 0$ in $\mathcal{N}_\delta$ and $\varphi \geq 0$ in $\mathcal{N}_\delta^c$ (thus, a transition through zero must be satisfied by φ on $\partial\mathcal{N}_\delta$).

Roughly speaking, and formally following the stochastic interpretation associated to analogous second-order problems discussed in [23], if we think of u as the value function of a stochastic optimal exit time problem with jumps, there is no incentive for the underlying random dynamics to stay in the interior of the domain because of the positive contribution of f. Then, trajectories starting from a point x close to the boundary have two choices: to exit immediately near x in a continuous fashion, or to jump outside $\bar{\Omega}$. In the latter case, since φ is positive "outside a neighborhood of the boundary" it discourages the trajectories to jump "too far". Therefore, the boundary data must be satisfied in the classical sense. Of course the function φ itself cannot be too oscillatory in order to prevent large payoff differences in nearby regions, from which condition (2.10) is natural.

We also point out that the assumption on φ in the above theorem can be weakened. Since neither continuity nor boundedness are needed in N_δ^c, we could just ask for an appropriate weighted $L^1(N_\delta^c, 1/(1+|x|)^{N+2s})$ bound.

In Sect. 5 we present a generalization of the previous theorem to a general $\mathcal{I}$ defined by linear operators whose symmetric kernels belong to $\mathcal{K}$ (with no equicontinuity assumption such as (2.5)). The theorem is in essence the same, though the range of p for which we can solve the problem differs. We refer to Theorem 5.1 for more details.

2.3 Perron's method and existence

We start remaking that Defintion 2.2 to the Dirichlet problem (1.1)–(1.3) is the same as in previous works which are going to be quoted here.

As in [3,30], to prove existence of a viscosity solution to our Dirichlet problem we consider a one-parameter family of continuous functions $\psi^k_\pm : \mathbb{R}^N \to \mathbb{R}$ such that $\psi^k_+ \geq \psi^k_-$ in $\mathbb{R}^N$ and such that $\psi^k_\pm = \varphi$ in Ω^c. In addition we take $\psi^k_\pm(x) \to \pm\infty$ as $k \to \infty$ for all $x \in \Omega$.

We extend f as a continuous function in $\mathbb{R}^N$ with the same L^∞ norm. For $x \in \mathbb{R}^N$, $q \in \mathbb{R}^N$, $l \in \mathbb{R}$, $\lambda \geq 0$ consider the degenerate elliptic operator

$$H_k(x, u, q, l) := \min\{u - \psi^k_-(x), \max\{u - \psi^k_+(x), \lambda u - l + |q|^p - f(x)\}\}$$

and let us consider the following obstacle problem

$$H_k(x, u(x), Du(x), \mathcal{I}(u, x)) = 0, \quad x \in \mathbb{R}^N, \tag{2.12}$$

where $\mathcal{I}$ is defined as in (2.3) for a general subclass $\{K_{a,b}\} \subset \mathcal{K}$.

For $O \subset \mathbb{R}^N$ open and regular denote

$$\Lambda_0(O) := \inf_{x \in O, K \in \mathcal{K}} \int_{O^c} \frac{(K(z) + K(-z))dz}{|x - z|^{N+2s}}.$$

For $x \in O$ and $K \in \mathcal{K}$ we see that the function

$$W_K(x) := \int_{O^c} \frac{(K(z) + K(-z))dz}{|x - z|^{N+2s}}$$

satisfies

$$2\gamma C_{N,s} \int_{O^c} \frac{dz}{|x - z|^{N+2s}} \leq W_K(x) \leq 2\Gamma C_{N,s} \int_{O^c} \frac{dz}{|x - z|^{N+2s}},$$

by (2.1). The function $x \mapsto \int_{O^c} \frac{dz}{|x-z|^{N+2s}}$ is strictly positive and continuous in O, and blows up on the boundary of O, from which we conclude that $\Lambda_0(O) \in (0, +\infty)$. We simply denote $\Lambda_0 = \Lambda_0(\Omega)$.

Now consider

$$\Psi_+(x) = (\Lambda_0(\mathcal{N}_1)^{-1} ||f||_\infty \eta(x) + ||\varphi||_\infty), \quad x \in \mathbb{R}^N,$$

where η is a smooth function such that $0 \leq \eta \leq 1$ with $\eta = 1$ in Ω and $\eta = 0$ in $\mathcal{N}_1^c$.

It can be seen that Ψ^+ is a viscosity supersolution to (2.12) for all k (note Ψ_+ does not depend on the L^∞ bounds of $\phi^k_\pm$) and analogously $\Psi_- = -\Psi_+$ is a viscosity subsolution to the problem. The problem (2.12) is degenerate elliptic in the whole space and following closely the arguments of Perron's method described in Proposition 1 in [2] leads to the existence of a solution $u_k \in C(\mathbb{R}^N)$ to (2.12). Moreover, the family of functions $\{u_k\}_k$ is uniformly bounded in $\mathbb{R}^N$, equal to φ in Ω^c for all k.

For $x \in \mathbb{R}^N$ denote

$$\bar{u}(x) = \limsup_{k\to\infty, y\to x} u_k(y); \quad \underline{u}(x) = \liminf_{k\to\infty, y\to x} u_k(y),$$

which are well defined for all $x \in \mathbb{R}^N$. We clearly have that $\bar{u} \geq \underline{u}$ in $\mathbb{R}^N$, $\underline{u} = \bar{u} = \varphi$ in $\bar{\Omega}^c$ and are respectively viscosity sub and supersolution to problem (1.1)–(1.3) in the sense of Definition 2.2. Thus, by regularity results up to the boundary given in [6] we have that $\bar{u}$ can be redefined as a Hölder continuous on $\bar{\Omega}$ (see Theorem 2.1 in [6]) which is sufficient to apply the strong comparison principle given in Theorem 3.2 of [8] to conclude that $\bar{u} \leq \underline{u}$ in $\bar{\Omega}$ and as a consequence, the existence and uniqueness of a viscosity solution $u = \bar{u} = \underline{u} \in C(\bar{\Omega})$ with generalized boundary conditions to (1.1)–(1.3). It also can be seen in [8] (Proposition 4.3) that $u \leq \varphi$ on $\partial\Omega$, but it is not guaranteed that $u = \varphi$ on $\partial\Omega$. However, again by strong comparison, it only suffices to construct a continuous subsolution to the Dirichlet problem which attains the boundary data to conclude the classical well-posedness.

Finally, we notice that the function $\Lambda_0^{-1}||f||_\infty \mathbf{1}_{\bar{\Omega}} + ||\varphi||_\infty$ is a supersolution to the Dirichlet problem (1.1)–(1.3), and a subsolution can be constructed in the same way. Applying strong comparison principle we conclude the estimate

$$|u(x)| \leq (\Lambda_0^{-1}||f||_\infty + ||\varphi||_\infty) \quad \text{for all } x \in \bar{\Omega}. \tag{2.13}$$

3 Technical lemmas

In this section we establish technical lemmas needed in order to conclude Theorem 2.3. Throughout this section we will focus on the family of equicontinuous bounded kernels $\bar{\mathcal{K}}$, see (2.5). It will be useful to consider the *extremal operators*

$$\mathcal{M}^+_{\bar{\mathcal{K}}}(u,x) := \sup_{K\in\bar{\mathcal{K}}} L_K(u,x); \quad \text{and} \quad \mathcal{M}^-_{\bar{\mathcal{K}}}(u,x) := \inf_{K\in\bar{\mathcal{K}}} L_K(u,x). \tag{3.1}$$

Note that we will simply write $\mathcal{M}^\pm$ whenever we have fixed the family of kernels, as is our case. Furthermore, notice that these operators have the same properties as the local Pucci's operator, in particular they are positive homogeneous and satisfy (see for example [13])

$$\mathcal{M}^-(u) \leq L_K(u) \leq \mathcal{M}^+(u) \quad \text{for all } K \in \bar{\mathcal{K}}, \tag{3.2}$$

$$\mathcal{M}^-(u-v) \leq \mathcal{I}(u) - \mathcal{I}(v) \leq \mathcal{M}^+(u-v), \tag{3.3}$$

$$\mathcal{M}^-(-u) = -\mathcal{M}^+(u). \tag{3.4}$$

Next lemma follows the lines of [26] (see also [27]), but we provide the proof for completeness.

Lemma 3.1 *Let Ω be a C^2 bounded domain in $\mathbb{R}^N$ and $s \in (0,1)$. There exists $\delta > 0$ such that for each $0 < \beta < s$ there exists $c_1 > 0$ satisfying*

$$\mathcal{M}^+(d(x)_+^\beta, x) \leq -c_1 d^{\beta-2s}(x) \quad \text{for } x \in \Omega_\delta.$$

The constant c_1 depends on β, N, s and $\mathcal{K}$ and is such that $c_1 \to 0$ as $\beta \to s^-$.

Proof We start with some preliminaries. For each $\beta \in (0, 2s)$ we define the function F by

$$F(\beta) = \int_{\mathbb{R}} \frac{(1+t)_+^\beta + (1-t)_+^\beta - 2}{|t|^{1+2s}} dt,$$

which is smooth, strictly convex and satisfies $F(0^+) < 0$. As it can be seen in [27] we have that $F(s) = 0$ and therefore we have $F(\beta) < 0$ for each $\beta \in (0, s)$. Consider now the constant

$$\tilde{C}(N,s) = \int_{\mathbb{R}^{N-1}} \frac{dy}{(|y|^2+1)^{(N+2s)/2}}.$$

We assert that the constant $c_1 > 0$ in the statement of the lemma is

$$c_1 = -F(\beta)\, \tilde{C}(N,s) \inf_{K \in \mathcal{K}} K(0). \tag{3.5}$$

Now we proceed with the proof. Notice that we can rewrite the inequality of the lemma as

$$d^{2s-\beta}(x)\mathcal{M}^+(d(x)_+^\beta) \leq -c_1 \quad \text{for } x \in \Omega_\delta.$$

Assume that the conclusion of the lemma does not hold. Then, there exists $\beta \in (0, s)$ and a sequence of points $x_n \in \Omega$ with $d(x_n) \to 0$ such that

$$\limsup_{n \to +\infty} d(x_n)^{2s-\beta} \mathcal{M}^+(d^\beta, x_n) > -c_1. \tag{3.6}$$

Let

$$l = \sup_{K \in \mathcal{K}} \int_{\mathbb{R}^N} \frac{(1+z_N)_+^\beta + (1-z_N)_+^\beta - 2}{|z|^{N+2s}} K(0) dz,$$

and we claim that

$$\limsup_{n \to +\infty} d(x_n)^{2s-\beta} \mathcal{M}^+(d^\beta, x_n) = l. \tag{3.7}$$

Suppose the claim holds, then a direct computation leads to

$$\int_{\mathbb{R}^N} \frac{(1+z_N)_+^\beta + (1-z_N)_+^\beta - 2}{|z|^{N+2s}} dz = F(\beta)\tilde{C}(N,s)$$

and therefore, from (3.7) and $F(\beta) < 0$ when $\beta \in (0, s)$, we deduce

$$\tilde{C}(N,s) F(\beta) \inf_{K \in \mathcal{K}} K(0) = \tilde{C}(N,s) \sup_{K \in \mathcal{K}} F(\beta) K(0) > -c_1.$$

The last inequality is a contradiction with the choice of c_1 in (3.5).

Now we prove the claim (3.7). We adopt the notation $d_n = d(x_n)$ and $\tilde{d}_n(z) = d(z)/d_n$ for each $n \in \mathbb{N}$ and $z \in \mathbb{R}^N$.

Using the homogeneity of the linear operators L_K we have

$$d_n^{2s-\beta}\mathcal{M}^+(d^\beta, x_n) = \sup_{K\in\mathcal{K}} \int_{\mathbb{R}^N} \frac{\delta(\tilde{d}_n^\beta, x_n, d_n z)}{|z|^{N+2s}} K(d_n z)dz.$$

Note now that the limit in (3.6) is finite. In fact, for each $K \in \mathcal{K}$ we split the previous integral as

$$I_1^n + I_2^n := \int_{B_1} \frac{\delta(\tilde{d}_n^\beta, x_n, d_n z)}{|z|^{N+2s}} K(d_n z)dz + \int_{B_1^c} \frac{\delta(\tilde{d}_n^\beta, x_n, d_n z)}{|z|^{N+2s}} K(d_n z)dz$$

Since $x_n \in \Omega$ for all n, we can perform a Taylor expansion of the function $\tilde{d}_n^\beta$ around x_n, which in addition to the universal upper bound for K leads us to

$$I_1^n \leq \Gamma\beta((1-\beta)2^{\beta-1} + 2^{\beta-1}d_n) \int_{B_1} |z|^{-N-2s+2}dz, \tag{3.8}$$

and therefore I_1^n is bounded above independently of n. For I_2^n we use the Lipschitz continuity of the distance function to write

$$\tilde{d}_n(x_n \pm d_n z) \leq 1 + |z|$$

for all $n \in \mathbb{N}$ and all $z \in \mathbb{R}^N$. Then, we have

$$I_2^n \leq \Gamma 2^{\beta+1} \int_{B_1^c} |z|^{-N-2s+\beta}dz, \tag{3.9}$$

from which we obtain the uniform boundedness of I_2^n since $\beta < s$. This concludes the finiteness of the limit in (3.6).

With some abuse of notation let $\{x_n\}$ be a subsequence in (3.7) realizing the limit, and fix $\epsilon > 0$. For each $n \in \mathbb{N}$ there exist $\bar{K} \in \bar{\mathcal{K}}$ depending on ϵ and n such that

$$\begin{aligned} &d_n^{2s-\beta}\mathcal{M}^+(d^\beta, x_n) - l \\ &\leq \epsilon/4 + \int_{\mathbb{R}^N} \tfrac{\delta(\tilde{d}_n^\beta, x_n, d_n z)}{|z|^{N+2s}} \bar{K}(d_n z)dz - \int_{\mathbb{R}^N} \tfrac{(1+z_N)_+^\beta + (1-z_N)_+^\beta - 2}{|z|^{N+2s}} \bar{K}(0)dz. \end{aligned}$$

Adding and subtracting the term

$$\int_{\mathbb{R}^N} \frac{\delta(\tilde{d}_n^\beta, x_n, d_n z)}{|z|^{N+2s}} \bar{K}(0)dz,$$

in the right-hand side of the above inequality, we arrive at

$$d_n^{2s-\beta}\mathcal{M}^+(d^\beta, x_n) - l \le \epsilon/4 + J_1^n + J_2^n, \tag{3.10}$$

where

$$J_1^n := \int_{\mathbb{R}^N} \frac{\delta(\tilde{d}_n^\beta, x_n, d_n z)}{|z|^{N+2s}} \left(\bar{K}(d_n z) - \bar{K}(0)\right) dz,$$
$$J_2^n := \int_{\mathbb{R}^N} \left(\delta(\tilde{d}_n^\beta, x_n, d_n z) - \left[(1+z_N)_+^\beta + (1-z_N)_+^\beta - 2\right]\right) \bar{K}(0)|z|^{-(N+2s)} dz.$$

Using the uniform boundedness of $\bar{K}(0)$ in terms of n and the asymptotic estimates given in [9] Lemma 4 we conclude that

$$|J_2^n| = o_n(1),$$

as $n \to \infty$.

Proceeding as in (3.9) and using the uniform boundedness of the family $\mathcal{K}$ we can consider $R_\epsilon > 1$ large enough to get

$$\int_{B_{R_\epsilon}^c} \frac{\delta(\tilde{d}_n^\beta, x_n, d_n z)}{|z|^{N+2s}} \left(\bar{K}(d_n z) - \bar{K}(0)\right) dz \le \epsilon/4.$$

Now, by assumption (2.5) we can write

$$\int_{B_{R_\epsilon}} \frac{\delta(\tilde{d}_n^\beta, x_n, d_n z)}{|z|^{N+2s}} \left(\bar{K}(d_n z) - \bar{K}(0)\right) dz \le m_\epsilon(R_\epsilon d_n) \int_{B_{R_\epsilon}} \frac{\delta(\tilde{d}_n^\beta, x_n, d_n z)}{|z|^{N+2s}} dz,$$

where $m_\epsilon = m_{R_\epsilon}$ is the modulus of continuity given in (2.5) and n has been taken large enough to have $d_n \le 1$. Similar arguments as in (3.8) and (3.9) let us conclude the existence of a constant $C_\epsilon > 0$ such that

$$\int_{B_{R_\epsilon}} \frac{\delta(\tilde{d}_n^\beta, x_n, d_n z)}{|z|^{N+2s}} \left(\bar{K}(d_n z) - \bar{K}(0)\right) dz \le C_\epsilon m_\epsilon(R_\epsilon d_n),$$

from which we conclude

$$J_1^n \le \epsilon/4 + C_\epsilon m_\epsilon(R_\epsilon d_n).$$

Replacing the estimates of J_1^n and J_2^n into (3.10) we get

$$d_n^{2s-\beta}\mathcal{M}^+(d^\beta, x_n) - l \le \epsilon/2 + o_n(1) + C_\epsilon m_\epsilon(R_\epsilon d_n).$$

Finally, taking n large in terms of ϵ allows us to get

$$d_n^{2s-\beta}\mathcal{M}^+(d^\beta, x_n) - l \le \epsilon.$$

Similar arguments provide us of an analogous reverse inequality which let us conclude (3.7). The proof is now complete. □

Remark 3.2 The above lemma explains the upper bound for p required in Theorem 2.3. The fact that the constant c_1 is strictly positive is crucial and this is possible because of $\beta < s$.

On the other hand, a balance of powers between the fractional Laplacian and the gradient term evaluated at the function d_+^β suggest a lower bound for β given by $(p-2s)/(p-1)$. Thus, in order to have such a number we require $(p-2s)/(p-1) < s$ and this is possible if $p < s/(1-s)$.

We also require the following bound for the fractional operator applied to power-type functions.

Lemma 3.3 *Let $\beta \in (0, 2s)$ and let $x_0 \in \mathbb{R}^N$. Then, there exists a constant $C_1 > 0$ such that*

$$\mathcal{M}^+(|\cdot - x_0|^\beta, x) \le C_1|x - x_0|^{\beta-2s} \quad \textit{for all } x \in \mathbb{R}^N \setminus \{x_0\}.$$

Moreover, there exists a constant $\bar{C}_1 > 0$ such that $C_1 \le \bar{C}_1$ as $s \to 1^-$.

Proof By translation invariance, we can assume $x_0 = 0$.

Let $K \in \bar{\mathcal{K}}$, $L = L_K$ as in (2.2) and denote $\rho(x) = |x|^\beta$. Using the notation from (2.4) we can write

$$L(\rho, x) = L[B_{|x|/2}(x)](\rho, x) + L[B_{|x|/2}(x)^c](\rho, x).$$

For the first term we perform a Taylor expansion with integral reminder to write

$$L[B_{|x|/2}](\rho, x) = \frac{1}{2}\int_0^1 (1-t)\int_{B_{|x|/2}} \langle D^2\rho(x+ty)y, y\rangle K(y)|y|^{-(N+2s)}dy\, dt.$$

A direct computation of the second derivative of the function ρ leads to the estimate

$$L[B_{|x|/2}](\rho, x) \le C|x|^{\beta-2} \int_{B_{|x|/2}} K(y)|y|^{2-(N+2s)}dy = C\,\Gamma|x|^{\beta-2s}.$$

where $C > 0$ depends only on β.

On the other hand, using that $|\rho(z) - \rho(y)| \le C|z-y|^\beta$ for each $y, z \in \mathbb{R}^N$ we have

$$L[B^c_{|x|/2}](\rho, x) \le C \int_{B_{|x|/2}(x)^c} |z|^{\beta-N-2s}K(z)dz \le C\,\Gamma|x|^{\beta-2s},$$

where $C > 0$ just depends on β. By adding the above estimates and taking supremum over $K \in \mathcal{K}$ we conclude the result. □

4 Proof of Theorem 2.3

The following lemma allows us to deal with the role of δ in Theorem 2.3.

Lemma 4.1 *Let $\delta > 0$ and consider f_φ^δ defined in (2.8). Then a viscosity solution u to problem (2.11) is a supersolution (in the sense of Definition 2.2) to the problem*

$$\begin{cases} \lambda u - \mathcal{I}(u) + |Du|^p = f_\varphi^\delta & \text{in } \Omega \\ u = \varphi \mathbf{1}_{\mathcal{N}_\delta} & \text{in } \Omega^c. \end{cases} \tag{4.1}$$

Proof We provide the proof in the context of classical solutions. The proof for viscosity solutions follows the same lines once the corresponding testing procedure is established.

Let u be a solution to (2.11) (satisfying $u = \varphi$ in $\bar{\Omega}^c$) and write $u = u_1 + u_2$ where $u_1 = u\mathbf{1}_{\mathcal{N}_\delta}$ and $u_2 = u\mathbf{1}_{\mathcal{N}_\delta^c}$.

Then, for each $x \in \Omega$ and $K \in \mathcal{K}$ we can write

$$\begin{aligned} L_K(u,x) &= L_K(u_1,x) + \int_{\mathbb{R}^N} \frac{u_2(x+z)}{|z|^{N+2s}}(K(z)+K(-z))dz \\ &= L_K(u_1,x) + \int_{\mathcal{N}_\delta^c - x} \frac{\varphi(x+z)}{|z|^{N+2s}}(K(z)+K(-z))dz, \end{aligned}$$

where the last equality comes from the definition of u_2. Then, by the monotoncity properties of the nonlinear operator $\mathcal{I}$ we get that

$$-\mathcal{I}(u,x) \le -\mathcal{I}(u_1,x) - \inf_{K\in\bar{\mathcal{K}}} \int_{\mathcal{N}_\delta^c - x} \frac{\varphi(x+z)}{|z|^{N+2s}}(K(z)+K(-z))dz.$$

Then, using this inequality, the fact that u is a (super)solution to (2.11), that $u_1 = u$ in Ω, and the definition of f_φ^δ we arrive at

$$\lambda u_1(x) - \mathcal{I}(u_1,x) + |Du_1(x)|^p \ge f_\varphi^\delta(x),$$

which means that u is a supersolution to (4.1). For $x \in \partial\Omega$ we follow the same idea by using Definition 2.2. □

From now on we will study this auxiliary problem, since a subsolution to (4.1) that agrees with the exterior data gives us the classical solvability to the original problem by comparison principle.

Now we present two results that are a direct consequence from the technical lemmas of the previous section.

Lemma 4.2 *Let $\Omega \subset \mathbb{R}^N$ a bounded domain, $\beta \in (0,2s)$ and let $y \in \partial\Omega$. Let $\delta > 0$ and consider the function*

$$\psi(x) = \psi_{y,\delta}(x) = |x-y|^\beta \mathbf{1}_{\mathcal{N}_\delta}(x), \quad x \in \mathbb{R}^N$$

with $\mathcal{N}_\delta$ defined above. Then

$$\mathcal{M}^+(\psi, x) \leq C_1|x-y|^{\beta-2s} \quad \textit{for each } x \in \Omega,$$

where $C_1 > 0$ is the constant in Lemma 3.3.

The proof of this result is a straight application of Lemma 3.3 and the fact that $\psi(x) \leq |x-y|^\beta$ for all $x \in \mathbb{R}^N$ with equality in Ω.

Lemma 4.3 *Let $\delta > 0$ and $A \in \mathbb{R}$. Then there exists universal constant $C_2 > 0$ such that*

$$\mathcal{M}^+(A1_{\mathcal{N}_\delta}, x) \leq C_2 A^-(d(x)+\delta)^{-2s} \quad \textit{for all } x \in \Omega,$$

where $A^- = \max\{0, -A\}$.

Moreover, there exists a constant $\bar{C}_2 > 0$ just depending on s, N such that $C_2 \leq \Gamma\bar{C}_2$.

Now we are in position to prove Theorem 2.3.

Proof of Theorem 2.3 (continuity up to the boundary) In view of the discussion given in Sect. 2, the result follows by constructing a subsolution $u \in C(\bar{\Omega})$ to (2.11) with $u = \varphi$ on $\partial\Omega$. Thanks to Lemma 4.1 we can focus on constructing such subsolution to (4.1) instead.

Let $\delta > 0$ and denote by M_δ the Hölder seminorm of φ in $\mathcal{N}_\delta$. For $y \in \partial\Omega$ we consider the function

$$u_y(x) = (\varphi(y) - M_\delta|x-y|^\beta)\mathbf{1}_{\mathcal{N}_\delta}(x) - \mu M_\delta d_+^\beta(x),$$

where $\mu > 1$ is a (large) constant independent of y to be fixed. Our goal is to prove that u_y is a subsolution to (4.1) in a $\bar{\Omega}$-neighborhood of $\partial\Omega$.

By (3.2) we can replace $\mathcal{I}$ by the extremal operator $\mathcal{M}^-$ in (4.1) to conclude the result.

Notice that $u_y = 0$ in $\mathcal{N}_\delta^c$, and for each $x \in \mathcal{N}_\delta \setminus \Omega$

$$u_y(x) \leq \varphi(y) - M_\delta|x-y|^\beta \leq \varphi(x),$$

from which we get that $u_y \leq \varphi\mathbf{1}_{\mathcal{N}_\delta}$ in Ω^c.

Let $x \in \Omega$. Appropriately using (3.2)–(3.4) and the positive homogeneity of the extremal operators, we can write

$$-\mathcal{M}^-(u_y, x) \leq \mu M_\delta \mathcal{M}^+(d_+^\beta, x) + M_\delta \mathcal{M}^+(|\cdot - y|^\beta \mathbf{1}_{\mathcal{N}_\delta}, x) + \mathcal{M}^+(-\varphi(y)\mathbf{1}_{\mathcal{N}_\delta}, x).$$

Then, applying Lemmas 3.1, 4.2 and 4.3 we get

$$-\mathcal{M}^-(u_y, x) \leq -c_1\gamma\mu M_\delta d(x)^{\beta-2s} + C_1\Gamma M_\delta|x-y|^{\beta-2s} + C_2\Gamma\varphi(y)^+(d(x)+\delta)^{-2s},$$

where c_1, C_1 and C_2 are universal positive constants and for $t \in \mathbb{R}$ we have written $t^+ = (-t)^- = \max\{0, t\}$.

Now, since $d(x) \le |x-y|$ we get

$$-\mathcal{M}^-(u_y, x) \le M_\delta d(x)^{\beta-2s}(-c_1\gamma\mu + C_1\Gamma) + C_2\Gamma\varphi(y)^+(d(x)+\delta)^{-2s}.$$

Thus, taking μ large in terms of the ratio $C_1\Gamma/(c_1\gamma)$ we can write

$$-\mathcal{M}^-(u_y, x) \le -c_1\mu\gamma M_\delta d(x)^{\beta-2s}/2 + C_2\Gamma||\varphi^+||_{L^\infty(\partial\Omega)}(d(x)+\delta)^{-2s}. \quad (4.2)$$

On the other hand

$$|Du_y(x)|^p \le M_\delta^p \beta^p |\mu d(x)^{\beta-1} Dd(x) + |x-y|^{\beta-2}(x-y)|^p,$$

for each $x \in \Omega$ and since $d(x) \le |x-y|$ we conclude that

$$|Du_y(x)|^p \le M_\delta^p d(x)^{\beta-2s}(1+\mu)^p.$$

Combining the above estimates for the nonlocal and gradient terms and using that $p(\beta-1) = \beta-2s$ we obtain

$$\begin{aligned}-\mathcal{M}^-(u_y, x) + |Du_y(x)|^p \le\ & M_\delta d(x)^{\beta-2s}\left(-c_1\mu\gamma/2 + M_\delta^{p-1}(1+\mu)^p\right)\\ &+ C_2\Gamma||\varphi^+||_{L^\infty(\partial\Omega)}(d(x)+\delta)^{-2s}.\end{aligned}$$

At this point we impose that M_δ, μ satisfy the relation

$$M_\delta\mu \le \bar{c}_p := (c_1\gamma/2^{p+2})^{1/(p-1)}, \quad (4.3)$$

which in conjunction to the previous inequality leads us to

$$\begin{aligned}-\mathcal{M}^-(u_y, x) + |Du_y(x)|^p \le\ & -c_1\mu\gamma M_\delta d(x)^{\beta-2s}/4\\ &+ C_2\Gamma||\varphi^+||_{L^\infty(\partial\Omega)}(d(x)+\delta)^{-2s}.\end{aligned}$$

Note that we can also pick μ, M_δ so that

$$\bar{c}_p/2 \le \mu M_\delta, \quad (4.4)$$

and therefore by fixing ϱ_0 such that

$$0 < \varrho_0 \le \left(\frac{16C_2\Gamma||\varphi^+||_{L^\infty(\partial\Omega)}}{\bar{c}_p c_1\delta^{2s}}\right)^{1/(\beta-2s)}, \quad (4.5)$$

we get the following inequality

$$-\mathcal{M}^-(u_y, x) + |Du_y(x)|^p \le -c_1\mu\gamma M_\delta d(x)^{\beta-2s}/8 \quad \text{for all } x \in \Omega_{\varrho_0}.$$

Since

$$u_y \le \inf_{\partial\Omega} \varphi + M_\delta \mathrm{diam}(\Omega)^\beta,$$

we conclude that

$$\begin{aligned} &\lambda u_y - \mathcal{M}^-(u_y, x) + |Du_y(x)|^p \\ &\le \lambda(\inf_{\partial\Omega} \varphi + M_\delta \mathrm{diam}(\Omega)^\beta) - \tilde{c}\mu M_\delta d(x)^{\beta-2s} \quad \text{for all } x \in \Omega_{\varrho_0}. \end{aligned}$$

where $\tilde{c} = c_1\gamma/8$.

Recall that $\beta < 2s$, therefore by taking ϱ_0 smaller in terms of $\lambda \inf_{\partial\Omega} \varphi$ and $\bar{c}_p$, and enlarging μ in terms of $\mathrm{diam}(\Omega)^\beta$ we obtain

$$\lambda u_y - \mathcal{M}^-(u_y, x) + |Du_y(x)|^p \le -\tilde{c}\mu M_\delta d(x)^{\beta-2s}/2 \quad \text{for all } x \in \Omega_{\varrho_0}. \tag{4.6}$$

In view of (4.4) we can consider ϱ_0 even smaller in terms of $\bar{c}_p$, $\tilde{c}$ and $||f_\varphi^\delta||_{L^\infty(\bar{\Omega})}$ (if necessary) to conclude that u_y is a classical (hence viscosity) subsolution to (4.1) in Ω_{ϱ_0}.

Consider now

$$\tilde{u}(x) = \sup_{y \in \partial\Omega} u_y(x), \ x \in \mathbb{R}^N. \tag{4.7}$$

By standard arguments, this function is a viscosity subsolution to (4.1) in Ω_{ϱ_0} in the sense of Definition 2.2, hence $\tilde{u} \le \varphi$ in Ω^c. Moreover, for each $x \in \partial\Omega$

$$\tilde{u}(x) \ge u_x(x) \ge \varphi(x),$$

which implies $\tilde{u}$ attains the boundary data in the classical sense.

Now assumption (2.9) implies that the function

$$\psi_-(x) = \varphi(x)1_{\mathcal{N}_\delta \setminus \bar{\Omega}}(x) + \inf_{\mathcal{N}_\delta \setminus \Omega}\{\varphi\} 1_{\bar{\Omega}}(x), \tag{4.8}$$

is a viscosity subsolution to (4.1) in Ω. In fact, the exterior and boundary inequalities are satisfied in the classical sense and for each $x \in \Omega$ we have

$$\begin{aligned} &\lambda\psi_-(x) - \mathcal{M}^-(\psi_-, x) + |D\psi_-(x)|^p \\ &\le \lambda \inf\nolimits_{\mathcal{N}_\delta \setminus \Omega}\{\varphi\} + \sup\nolimits_{K \in \bar{\mathcal{K}}} \int_{\Omega^c - x} \frac{\inf_{\mathcal{N}_\delta \setminus \Omega}\{\varphi\} - \varphi \mathbf{1}_{\mathcal{N}_\delta}(x+z)}{|z|^{N+2s}} (K(z) + K(-z))dz \\ &\le \lambda \inf\nolimits_{\mathcal{N}_\delta \setminus \Omega}\{\varphi\} + \sup\nolimits_{K \in \bar{\mathcal{K}}} \int_{\mathcal{N}_\delta^c - x} \frac{\inf_{\mathcal{N}_\delta \setminus \Omega}\{\varphi\}}{|z|^{N+2s}} (K(z) + K(-z))dz \\ &\le (\lambda + \lambda_{0,+}^\delta(x)) \inf\nolimits_{\mathcal{N}_\delta \setminus \Omega}\{\varphi\}, \quad (\text{resp. } \ \lambda_{0,-}^\delta(x) \text{ if } \inf\nolimits_{\mathcal{N}_\delta \setminus \Omega}\{\varphi\} < 0), \end{aligned}$$

where we have used the definition of $\lambda_{0,\pm}^\delta(x)$ given in (2.6) and (2.7). By (2.9) we conclude that ψ_- is a subsolution to (4.1) in Ω.

Finally, by standard arguments the function

$$\underline{u}(x) := \max\{\tilde{u}(x), \psi_-(x)\} \tag{4.9}$$

is the required viscosity subsolution attaining the boundary data, provided $\tilde{u}(x) \leq \psi_-(x)$ for all $x \in \Omega$ such that $d(x) = \varrho_0$. Notice that for such points

$$\tilde{u}(x) \leq \sup_{\partial\Omega} \varphi - M_\delta(1+\mu)\varrho_0^\beta,$$

and

$$\psi_-(x) = \inf_{\mathcal{N}_\delta \setminus \Omega} \{\varphi\}.$$

Therefore to get $\tilde{u}(x) \leq \psi_-(x)$ it is sufficient to find μ large enough so that

$$\sup_{\mathcal{N}_\delta} \varphi - \inf_{\mathcal{N}_\delta} \varphi \leq M_\delta(1+\mu)\varrho_0^\beta.$$

In view of (2.10), the previous inequality holds as long as μ satisfies

$$(\mathrm{diam}(\mathcal{N}_\delta)\varrho_0^{-1})^\beta \leq \mu.$$

Thus, fixing μ large as required above (in terms of the data and ϱ_0) and M_δ small in order to have (4.3), (4.4) concludes the proof. □

Corollary 4.4 *Assume hypotheses of Theorem 2.3 hold. Then, there exist $M, \varrho > 0$ just depending on the data such that the unique viscosity solution u to problem (2.11) satisfies*

$$|u(x) - u(\hat{x})| \leq M d^\beta(x), \quad \textit{for all } x \in \Omega_\varrho,$$

where $\hat{x} \in \partial\Omega$ satisfies $d(x) = |x - \hat{x}| \leq \rho$.

Proof From the previous proof we have that for all x close to the boundary

$$u(x) \geq \tilde{u}(x),$$

where $\tilde{u}$ is given by (4.7). Since $\tilde{u}(x) \geq u_{\hat{x}}(x)$, then we conclude that

$$u(x) - u(\hat{x}) \geq -Md(x)^\beta,$$

for some $M > 0$ just depending on the data.

For $y \in \partial\Omega$ define

$$v_y(x) = \left(\varphi(y) + \tilde{M}|x-y|^{\beta^*}\right) \mathbf{1}_{\mathcal{N}_\delta}(x) + 2\tilde{M} d_+^{\beta^*}(x), \; x \in \mathbb{R}^N,$$

with β^* as in Theorem 2.3 and $\tilde{M} > 0$ large. A similar computation as before leads us to the following bounds for $x \in \Omega$ close to the boundary

$$\mathcal{M}^+(u_y, x) \leq C\tilde{M}d^{\beta^*-2s}(x), \quad \text{and} \quad |Du_y(x)| \geq c\tilde{M}d^{\beta^*-1}(x),$$

for some $C, c > 0$ depending only on the data. Given that $p > 2s > 1$, we can take $\tilde{M}$ large enough so that v_y satisfies

$$\lambda v_y - \mathcal{I}(v_y) + |Dv_y| \geq f \quad \text{in } \Omega_\delta,$$

for some $\rho > 0$ fixed small. Then, defining

$$\tilde{v}(x) = \inf_{y \in \partial\Omega} v_y(x),$$

and taking $\tilde{M}$ large in terms of $||u||_\infty$, δ and the data, we obtain that $\tilde{v} \geq u$ in Ω_ρ^c. Then, by the strong comparison principle (see [8]) we get

$$u(x) - u(\hat{x}) \leq Md(x)^\beta,$$

from which the result follows. □

4.1 Regularity

Interior Hölder regularity (with exponent β^*) can be obtained as in [6]. Using Corollary 4.4 we can extend this result up to the boundary.

Proof of Theorem 2.3. (Regularity) Let $u \in C(\mathbb{R}^N)$ be the unique viscosity solution to problem (2.11) and fix $x_0 \in \Omega$. We assume that $d(x_0) \leq \varrho/8$ with $\varrho > 0$ as in Corollary 4.4. The general case $x_0 \in \Omega$ follows the same ideas as the ones presented below.

Denote by $\hat{x}_0$ the projection of x_0 to $\partial\Omega$ and for $L > 0$ consider the function

$$\Phi : x \mapsto u(x) - u(x_0) - L|x - x_0|^\beta, \quad x \in \mathbb{R}^N.$$

We claim that, for L large, just in terms of the data but not on $d(x_0)$, Φ is nonpositive in Ω. From this, we conclude that

$$u(x) - u(x_0) \leq L|x - x_0|^\beta \quad \text{for all } x \in \Omega,$$

and since L does not depend on x_0, interchanging the role of x_0 we conclude the Hölder regularity up to the boundary.

Let us prove the claim. By contradiction, we assume the opposite, that is $\sup_{\mathbb{R}^N} \Phi > 0$. With an initial choice $L \geq 8||u||_\infty \varrho^{-1}$ (recall the a priori bounds (2.13), which implies L just depend on the data), we get that the supremum of Φ in $\mathbb{R}^N$ is achieved at some point $\bar{x} \in \bar{B}_\varrho(\hat{x}_0)$, and since the maximum is strictly positive we get that $\bar{x} \neq x_0$.

We will prove that $\bar{x} \in \Omega$. Suppose not, then

$$u(\bar{x}) - u(x_0) = u(\bar{x}) - u(\hat{x}_0) + u(\hat{x}_0) - u(x_0) \leq \varphi(\bar{x}) - \varphi(\hat{x}_0) + Md(x_0)^\beta$$

where the last inequality comes from Corollary 4.4. Since φ is β Hölder continuous we see that

$$u(\bar{x}) - u(x_0) \leq M(|\bar{x} - \hat{x}_0|^\beta + d(x_0)^\beta)$$

for some $M > 0$ large enough. It is direct to check that $d(x_0) \leq |\bar{x} - x_0|$ and that $|\bar{x} - \hat{x}_0| \leq 4|x_0 - \bar{x}|$, and therefore we arrive at

$$u(\bar{x}) - u(x_0) \leq CM|\bar{x} - x_0|^\beta,$$

for some $C > 0$. Then, taking $L > 0$ large enough in terms of the data we conclude that

$$0 < u(\bar{x}) - u(x_0) - L|\bar{x} - x_0|^\beta \leq 0$$

which is a contradiction, and therefore $\bar{x} \in \Omega$.

Since $\bar{x} \in \Omega$ and $\bar{x} \neq x_0$, we can use the function $x \mapsto \phi(x) := L|x - x_0|^\beta$ as a test function (regarded as a subsolution to the problem) at the point $\bar{x}$ to deduce

$$-\mathcal{I}[B_\delta](\phi, \bar{x}) - \mathcal{I}[B_\delta^c](u, \bar{x}) + |D\phi(\bar{x})|^p \leq f(\bar{x}),$$

for all $0 < \delta \leq \varrho/4$.

Now, $D\phi(\bar{x})$ can be explicitly computed and since $\mathcal{I} \leq \mathcal{M}^+$ we can apply Lemma 4.2 to get

$$-C_1 L|\bar{x} - x_0|^{\beta - 2s} - C_2||u||_\infty \varrho^{-2s} + L^p|\bar{x} - x_0|^{p(\beta-1)} \leq ||f||_\infty,$$

for some $C_1, C_2 > 0$ just depending on the data.

Given that $p(\beta - 1) = \beta - 2s$ the last inequality can be rewritten as

$$L^p|\bar{x} - x_0|^{p(\beta-1)}(-C_1 L^{1-p} + 1) \leq ||f||_\infty + C_2||u||_\infty \varrho^{-2s}.$$

Since $p > 2s > 1$ we can take L large in terms of C_1 and p to get

$$L^p|\bar{x} - x_0|^{p(\beta-1)}/2 \leq (||f||_\infty + C_2||u||_\infty \varrho^{-2s}),$$

which drives to a contradiction by taking L large enough. □

5 Remarks and extensions

In this section we extend our results to the general family of kernels $\mathcal{K}$ and make some comments regarding the robustness of our result.

5.1 Extension to general nonlinear operators

We start this subsection by extending our results to the nonlinear nonlocal operators $\mathcal{I}$ as in (2.3) with kernels in the general class $\mathcal{K}$ satisfying (2.1) and with symmetric kernels.

As in (3.1) we consider the extremal operators associated to this class $\mathcal{K}$, which we will simply denote $\mathcal{M}^{\pm}$.

The aim is to provide the main strategy to get the following existence and regularity theorem concerning Dirichlet problems with exterior condition. For simplicity we will assume that the boundary data is compactly supported. In the general case we can proceed as before by cutting the data and studying an auxiliary equivalent problem.

In what follows, we denote $\beta^* = (p-2s)/(p-1)$ as in Theorem 2.3.

Theorem 5.1 *Let $s \in (1/2, 1)$, $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^2 boundary, $\lambda \geq 0$ and $f \in C(\bar{\Omega})$. Let $\mathcal{I}$ as in (2.3) defined through an arbitrary subfamily of $\mathcal{K}$. Assume $\varphi \in C(\Omega^c)$ is compactly supported and denote $D = \mathrm{supp}\{\varphi\}$. For $x \in \Omega$ define*

$$\lambda_{0,+}(x) = 2\Gamma \int_{D^c - x} \frac{dz}{|z|^{N+2s}}, \quad \text{and} \quad \lambda_{0,-}(x) = 2\gamma \int_{D^c - x} \frac{dz}{|z|^{N+2s}} \tag{5.1}$$

and assume $\inf_{D\setminus\Omega}\{\varphi\} > 0$ (resp. $\inf_{D\setminus\Omega}\{\varphi\} < 0$)

$$\inf_{D\setminus\Omega}\{\varphi\} \leq \inf_{\Omega}\left\{(\lambda + \lambda_{0,+})^{-1} f\right\}. \tag{5.2}$$

(resp. replacing $\lambda_{0,+}$ by $\lambda_{0,-}$).

Then there exists $s^+ \in (0, s)$ and $M_0 > 0$ depending on the data such that for each $p \in (2s, \frac{2s - s_+}{1 - s_+})$ and φ satisfying

$$|\varphi(x) - \varphi(y)| \leq M|x - y|^{\beta^*} \quad \text{for } x, y \in D \setminus \Omega, \tag{5.3}$$

for some $M \leq M_0$, the Dirichlet problem

$$\begin{cases} \lambda u - \mathcal{I}(u, x) + |Du|^p = f(x) & \text{in } \Omega \\ u = \varphi & \text{in } \Omega^c, \end{cases} \tag{P'}$$

has a unique viscosity solution $u \in C(\mathbb{R}^N)$.

Moreover, the restriction of u to $\bar{\Omega}$ belongs to $C^{\beta^*}(\bar{\Omega})$.

We will present the technical lemmas proved in Sect. 3 just highlighting the main differences in the current setting. The main difference between Theorems 2.3 and 5.1 is the restriction on the power profile p of the nonlinearity of the gradient in the latter result. This restriction is a consequence of the following lemma.

Lemma 5.2 *Let Ω be a C^2 bounded domain in $\mathbb{R}^N$ and $s \in (0,1)$. Denote by $\mathcal{M}^+$ the maximal extremal operator associated to the class $\mathcal{K}$. Them ,there exist $s_+ \in (0,s)$ such that, for all $\beta \in (0, s_+)$*

$$\mathcal{M}^+(d(x)_+^\beta) \leq -c_1 d^{\beta-2s}(x) \qquad \text{for all } x \in \Omega_\delta,$$

with $c_1,\ \delta > 0$.

The proof of this result follows the lines of Lemma 3.1. The main difference is the computation of the scaled limit given by (3.7). In that lemma, this is a consequence of the equicontinuity assumption 2.5. This time, the symmetry of the kernels leads to the following identity

$$\mathcal{M}^+(u,x) = C_{N,s} \int_{\mathbb{R}^N} S^+(\delta(u,x,y))|y|^{-(N+2s)}dy, \tag{5.4}$$

where for given $t \in \mathbb{R}$ we denote

$$S^+(t) = \Gamma t_+ - \gamma t_-,$$

see [13] for a proof of this fact. This expression is compatible with dominated convergence theorems and therefore the limit in (3.7) is replaced in the current setting by

$$\lim_{n\to\infty} d(x_n)^{2s-\beta}\mathcal{M}^+(d_+^\beta, x_n) = C\int_{\mathbb{R}} S^+(\delta(t_+^\beta, 1, y))|y|^{-(1+2s)}dy, \tag{5.5}$$

where $C = C(N,s) > 0$ is a constant, see [9] for a proof of this fact for a similar family of nonlinear operator.

As it can be seen in the proof of Theorem 2.3, the sign of the right-hand side in the inequality given by Lemma 5.2 is crucial to get the conclusion of Theorem 5.1 and in the current setting, the same sign disposition can be obtained provided $\beta < s_+$ for certain $0 < s_+ \leq s$ which is proper of the family of kernels $\mathcal{K}$. This also determines the new upper bound of p since this restriction coupled with the natural condition on the exponent $\beta = (p-2s)/(p-1)$ gives the condition $p \in (2s, \frac{2s-s_+}{1-s_+})$.

The second ingredient is the analogous of Lemma 3.3 but its proof is direct thanks to ellipticity of the class $\mathcal{K}$. Once the two technical lemmas are available, the construction of the barrier u in (4.9) follows with minor changes.

Remark 5.3 We point out that s_+ and λ_0 are independent of the operator $\mathcal{I}$ and therefore the result is not "sharp", since the main ingredients are not associated with the corresponding operator. This is to be expected in the generality the result is presented.

If we consider a subclass of $\mathcal{K}$ with certain assumptions we can get "sharper results". For example, consider

$$\mathcal{I}(u,x) = \inf_{a\in A}\sup_{b\in B}\int_{\mathbb{R}^N} \frac{\delta(u,x,z)}{|z|^{N+2s}} k_{a,b}(\hat{z})dz,$$

where $\hat{z} = z/|z|$ for each $z \neq 0$ and $k_{a,b} \colon \mathbb{S}^{N-1} \to \mathbb{R}$ is symmetric and uniformly bounded above and below, among other technical assumptions (see [20] for further properties of the operator).

In this case, an analogous result to Lemma 3.1 can be stated but with the restriction $0 < \beta < s_{\mathcal{I}}$ for some $s_+ < s_{\mathcal{I}} < s$. That is, we can precisely compute the allowed upper bound $(2s - s_{\mathcal{I}})/(1 - s_{\mathcal{I}})$ for each nonlinear operator I within the family of kernels appearing in [20].

5.2 Limit as $s \to 1$

Let $\lambda > 0$, $p > 2$, $f \in C(\bar{\Omega})$, $\varphi \in C(\partial\Omega)$ and consider the second-order Dirichlet problem

$$\begin{cases} \lambda u - \Delta u + |Du|^p = f \text{ in } \Omega \\ \qquad\qquad\qquad u = \varphi \text{ on } \partial\Omega. \end{cases} \tag{5.6}$$

As it is stated in [14], this problem is solvable by a unique viscosity solution $u \in C(\bar{\Omega})$ attaining the boundary data provided

$$\lambda \inf_{\partial\Omega} \varphi \leq \inf_{\Omega} f,$$

and φ additionally satisfies

$$|\varphi(x) - \varphi(y)| \leq M|x - y|^{(p-2)/(p-1)},$$

for some M small enough in terms of the data.

In view of the definition of f_φ^δ given in (2.8), notice that there exists a constant $C > 0$ not depending on δ nor s such that

$$||f - f_\varphi^\delta||_{L^\infty(\bar{\Omega})} \leq C C_s \delta^{-2s}.$$

Then we have $f_\varphi^\delta \to f$ uniformly in $\bar{\Omega}$ if $\delta \to \infty$, but also if $s \to 1^-$ when $\delta > 0$ is fixed. This fact is important to obtain robust estimates as $s \to 1^-$, therefore the conditions for the second-order problem (5.6) can be obtained by a passage to the limit in equation (1.1)–(1.3).

More precisely, since the domain is smooth, we can extend the boundary data φ in (5.6) to Ω^c as a C^β function, which we still denote by φ. Thus, in the passage to the limit $s \to 1^-$ we have compactness of the family of solutions when $\lambda > 0$. Now, the normalizing constant $C_{N,s}$ in the fractional Laplacian vanishes as $s \to 1$ and therefore λ_0 vanishes. Finally note that the inequalities (1.6)–(1.7) are independent of δ in the limit, leading to the analogous conditions for the local problem above by taking $\delta \to 0$.

Notice that in view of Theorems 2.3 and Theorem 5.1 we are able to provide ad-hoc conditions for solvability of fully nonlinear second-order problems.

For example, following the same ideas developed above, it is possible to get conditions for x-dependent second-order operators of the form $\mathrm{Tr}(a(x)D^2u(x))$, as in [14]. This is accomplished by taking the limit on nonlocal problems of the form

$$L(u,x)=C_{N,s}\int_{\mathbb{R}^N}\delta(u,x,z)|\sigma(x)^T z|^{-(N+2s)}dz,$$

where $\sigma:\bar{\Omega}\to S^N_+$ is a continuous matrix valued function with

$$\gamma I_N\leq a(x)\leq \Gamma I_N \quad \text{for all } x\in\bar{\Omega},$$

where I_N is the identity matrix.

Along the same line, it is possible to get results for fully nonlinear operators of the form

$$F(D^2u(x))=\inf_i \sup_j(\mathrm{Tr}(A_{ij}(x))D^2u(x))$$

where A is a symmetric matrix satisfying $\gamma I_N\leq A_{ij}(x)\leq \Gamma I_N$.

5.3 Open questions

We have left some open problems regarding the existence of viscosity solutions attaining the boundary condition in the cases $s=1/2$ and $p=s/(1-s)$ or $p=\frac{2s-\beta_1}{1-\beta_1}$. Each one of them represent extremal cases that are not present in the local case. The case $p=s/(1-s)$ is of special interest since the standard barrier (a power of the distance) cannot be used and therefore a different strategy has to be employed.

Another interesting open question comes from the regularity view point. In [6] it is proven that the solution to (1.1) is $C^\sigma(\bar{\Omega})$ with $\sigma=(p-2s)/p$ regardless the solution attains the boundary condition. On the other hand, the (global) Hölder exponent $\beta=(p-2s)/(p-1)$ obtained in this paper is a direct consequence from the fact that the boundary data is attained. Hence a natural question would be the a priori C^β Hölder estimates for solutions in this general framework, regardless whether they attain the boundary data or not.

Acknowledgments We would like to thank the referee for the careful reading of this manuscript, which lead to several improvements in the presentation of our results. G. Dávila was partially supported by Fondecyt Grant No. 11150880. A. Q. were partially supported by Fondecyt Grant No. 1151180, Programa Basal, CMM. U. de Chile and Millennium Nucleus Center for Analysis of PDE NC130017. E. T. was partially supported by Fondecyt Postdoctoral Grant No. 3150100 and Conicyt PIA Grant No. 79150056.

References

1. Alarcón, S., García-Melián, J., Quaas, A.: Existence and non-existence of solutions to elliptic equations with a general convection term. Proc. R. Soc. Edinburgh Sec A **144**(2), 225–239 (2014)
2. Alvarez, O., Tourin, A.: Viscosity solutions of nonlinear integro-differential equations. Annales de L'I.H.P., section C 13(3):293–317 (1996)
3. Barles, G., Chasseigne, E., Imbert, C.: On the Dirichlet Problem for Second Order Elliptic Integro-Differential Equations Indiana U. Math, Journal (2008)
4. Barles, G., Da Lio, F.: On the generalized Dirichlet problem for viscous Hamilton-Jacobi Equations. J. Math. Pures et Appl. **83**(1), 53–75 (2004)

5. Barles, G., Imbert, C.: Second-order elliptic integro-differential equations: viscosity solutions' theory revisited. IHP Anal. Non Linéare **25**(3), 567–585 (2008)
6. Barles, G., Koike, S., Ley, O. and Topp, E.: Regularity results and large time behavior for integro-differential equations with coercive Hamiltonians. Calc. Var. Partial Differ. Eq. (2014). doi:10.1007/s00526-014-0794-x
7. Barles, G., Perthame, B.: Exit time problems in optimal control and vanishing viscosity method. SIAM J. Control Optim. **26**, 1133–1148 (1988)
8. Barles, G., Topp, E.: Existence, uniqueness and asymptotic behavior for nonlocal parabolic problems with dominating gradient terms. SIAM J. Math. Anal. **48**(2), 1512–1547 (2016)
9. Barrios, B., Del Pezzo, L., García-Melián, J., Quaas, A.: A priori bounds and existence of solutions for some nonlocal elliptic problems (**Preprint**)
10. Biswas, I.H.: On zero-sum stochastic differential games with jump-diffusion driven state: a viscosity solution framework. SIAM J. Control Optim 50(4):1823–1858
11. Bogdan, K., Burdzy, K., Chen, Z.-Q.: Censored stable processes. Prob. Theory Rel. Fields **127**(1), 89–152 (2003)
12. Buckdahn, R., Hu, Y., Li, J.: Stochastic representation for solutions of Isaacs type integralpartial differential equations. Stoch. Process. Appl. **121**(12), 2715–2750 (2011)
13. Caffarelli, L., Silvestre, L.: Regularity theory for nonlocal integro-differential equations. Comm. Pure Appl. Math **62**(5), 597–638 (2009)
14. Capuzzo-Dolcetta, I., Leoni, F., Porretta, A.: Hölder estimates for degenerate elliptic equations with coercive Hamiltonians. Trans. Am. Math. Soc. **362**(9), 4511–4536 (2010)
15. Chasseigne, E.: The Dirichlet problem for some nonlocal diffusion equations. Differ. Integral Equ. **20**(12), 1389–1404 (2007)
16. Crandall, M.G., Ishii, H., Lions, P.-L.: User's guide to viscosity solutions of second order partial differential equations. Bull. Am. Math. Soc. (N.S.) **27**(1), 1–67 (1992)
17. Da Lio, F.: Comparison Results for quasilinear equations in annular domains and applications. Commun. Partial Differ. Equ. **27**(1 & 2), 283–323 (2002)
18. Di Neza, E., Palatucci, G., Valdinoci, E.: Hitchhiker's guide to the fractional Sobolev spaces. Bull. Sci. Math. **136**(5), 521–573 (2012)
19. Felmer, P., Quaas, A.: Fundamental solutions and Liouville type theorems for nonlinear integral operators. Adv. Math. **226**(3), 2712–2738 (2011)
20. Felmer, P., Quaas, A.: Fundamental solutions for a class of Isaacs integral operators. Discrete Contin. Dyn. Syst. **30**(2), 493508 (2011)
21. Felmer, P., Topp, E.: Uniform equicontinuity for a family of zero order operators approaching the fractional Laplacian. Commun. Partial Differ. Equ. **40**(9), 1591–1618 (2015)
22. Fleming, W., Soner, H.: Controlled Markov processes and viscosity solutions applications of mathematics. Springer-Verlag, New York (1993)
23. Lasry, J.M., Lions, P.L.: Nonlinear elliptic equations with singular boundary conditions and stochastic control with state constraints. Math. Ann. **283**, 583–630 (1989)
24. Gilbarg, D., Trudinger, N.S.: Elliptic partial differential equations of second order. Springer-Verlag, Berlin (2001)
25. Kawohl, B., Kutev, N.: A study on gradient blow up for viscosity solutions of fully nonlinear, uniformly elliptic equations. Act. Math. Scientia 32 B (1):15–40 (2012)
26. Quaas, A., Salort, A.: Principal eigenvalue of a integro-differential elliptic equation with a drift term, in preparation
27. Ros-Oton, X., Serra, J.: The Dirichlet problem for the fractional Laplacian: Regularity up to the boundary. Journal de Mathèmatiques Pures et Appliquées **101**(3), 275–302 (2014)
28. Soner, H.M.: Optimal: control problems with state-space constraints. SIAM J. on Control and Optimization 24, Part I: pp 552–562 II, 1110–1122 (1986)
29. Tchamba, T.T.: Large time behavior of solutions of viscous hamilton-jacobi equations with superquadratic Hamiltonian. Asymptot. Anal. **66**, 161–186 (2010)
30. Topp, E.: Existence and uniqueness for integro-differential equations with dominating drift terms. Commun. Partial Differ. Equ. **39**(8), 1523–1554 (2014)