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On critical exponents for Lane–Emden–Fowler-type equations with a singular extremal operator

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Abstract In this article, we consider the nonlinear elliptic equation

$$|\nabla u|^{\beta}\mathcal{M}^{+}{}_{\lambda,\Lambda}(D^2u) + u^p = 0 \text{ in } \mathbb{R}^N. \tag{$\star$}$$

Here, $\mathcal{M}^{+}{}_{\lambda,\Lambda}$ denotes Pucci's extremal operator with parameters $\Lambda \geq \lambda > 0$ and $-1 < \beta < 0$. We prove the existence of a critical exponent p_+^* that determines the range of $p > 1$ for which we have the existence or nonexistence of a positive radial solution to ($\star$). In addition, we describe the solution set in terms of the parameter p and find two new critical exponents $1 < p_+^* < \tilde{p}_\beta$ for the equation ($\star$), where the solution set sharply changes its qualitative properties when the value of p exceeds these critical exponents.

Keywords Critical exponent · Singular extremal operator · Emden–Fowler transformation

Mathematics Subject Classification 35J60 · 35J75

1 Introduction

The Lane–Emden–Fowler equation is one of the simplest nonlinear elliptic partial equations (see [1,2])

$$\Delta u + u^p = 0 \quad \text{in} \quad \mathbb{R}^N. \tag{1.1}$$

Understanding this equation requires various important tools, such as the Emden–Fowler transformation, the Pohozaev identity, energy integrals, moving plane techniques, the Kevin

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transform and the Harnack inequalities. All these tools can be used to prove various basic results for Eq. (1.1) or for more general equations.

This equation has a solution set with a structure that strongly depends on the exponent p. One of the main results for Eq. (1.1) is that the number $p_N^* = \frac{N+2}{N-2}$ is the the principal critical exponent, that is:

- If $1 < p < p_N^* = \frac{N+2}{N-2}$, then there is no nontrivial positive solution to Eq. (1.1) (see the result by Gidas and Spruck in [3] and also the paper Chen and Li [4]).
- If $p = p_N^*$, then Eq. (1.1) possesses exactly one positive solution, up to scaling and translation. This solution additionally satisfies $u(|x|)|x|^{N-2} \to C$ as $|x| \to \infty$ (see the paper by Caffarelli, Gidas and Spruck [5]).
- If $p > p_N^*$, then Eq. (1.1) has a radial positive solutions that behave like $C|x|^{-\alpha}$ near infinity, where $\alpha = \frac{2}{p-1}$ and $C > 0$.

The number p_N^* is known as the critical Sobolev exponent, since the Sobolev conjugate of 2 is $2^* := 2N/(N-2) = p_N^* + 1$. Recall that 2^* appears for example in compact embedding. In fact, if U is a bounded open subset of $\mathbb{R}^N$ and ∂U is C^1 then $H^1(U)$ is compactly embedded in $L^p(U)$ for $1 \leq p < 2^*$ (see for example Theorem 1 in Section 5.7 of [6]).

When $1 < p \leq \frac{N}{N-2} := p^s$, then a Liouville-type theorem or nonexistence result can be determined for classical supersolution of (1.1); that is, for:

$$\Delta u + u^p \leq 0 \quad \text{in} \quad \mathbb{R}^N. \tag{1.2}$$

This exponent is optimal in the sense that the solution of (1.2) exists if $p > p^s$; this number is sometimes called the second critical exponent of (1.1). The results for the supersolution are described by Gidas in [7].

This last result can be extended by replacing the Laplacian with a fully nonlinear elliptic operator; for example, Pucci's extremal operator defined as:

$$\begin{gathered} \mathcal{M}^+{}_{\lambda,\Lambda}(M) = \Lambda \sum\nolimits_{e_\imath > 0} e_\imath + \lambda \sum\nolimits_{e_\imath < 0} e_\imath, \\ \text{where } e_1, \ldots, e_N \quad \text{are the eigenvalues of M} \end{gathered} \tag{1.3}$$

(see, for example, [8] and [9] for a similar definition).

In [10], the authors Cutri and Leoni studied the following inequality

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2 u) + u^p \leq 0, \quad \text{in} \quad \mathbb{R}^N. \tag{1.4}$$

They proved that for $1 < p \leq \tilde{p}^s = \frac{\tilde{N}}{\tilde{N}-2}$, Eq. (1.4) has no nontrivial positive viscosity solution, where the dimension-like number $\tilde{N}$ is define by

$$\tilde{N} = \frac{\lambda}{\Lambda}(N-1) + 1. \tag{1.5}$$

For $p > \tilde{p}^s$, they also presented a solution of the inequality, showing that their exponent is optimal.

In [11], Birindelli and Demengel extended the aforementioned results to more general inequality of the form:

$$|\nabla u|^\beta \mathcal{M}^+{}_{\lambda,\Lambda}(D^2 u) + u^p \leq 0 \quad \text{in} \quad \mathbb{R}^N, \tag{1.6}$$

with $\beta \in (-1, +\infty)$. The authors derived the existence of a critical exponent $\tilde{p}^b = \frac{1+(\beta+1)(\tilde{N}-1)}{\tilde{N}-2}$ such that for $0 < p \leq \tilde{p}^b$, there is no nontrivial positive viscosity solution for (1.6).

These results for the supersolution were extended to others different operators (see, for example, [12,13], and [14] and the references therein).

The problem for equation

$$\mathcal{M}^{+}{}_{\lambda,\Lambda}(D^2u) + u^p = 0 \quad \text{in} \quad \mathbb{R}^N, \tag{1.7}$$

with $\Lambda \neq \lambda$ is more complex than the case when the differential operator is the Laplacian [i.e., When $\Lambda = \lambda = 1$, then (1.2) is (1.7)]. This is because the natural candidate for the critical exponent, $(\tilde{N}+2)/(\tilde{N}-2)$, is not the critical exponent for existence and nonexistence.

In fact, in [15], Felmer and the second author studied radially symmetric positive solution of (1.7) and proved the existence of a critical exponent p^* that corresponds to the critical exponent p_N^* for the Laplacian, which separates the range in p for existence from nonexistence of solution to (1.7). They also proved that the critical exponent p^* satisfies the bounds $\max\{\frac{\tilde{N}}{\tilde{N}-2}, p_N^*\} < p^* < \frac{\tilde{N}+2}{\tilde{N}-2}$. These results were generalized in [16]. Note that the result for nonexistence of nonradially symmetric solution still remains unsolved.

The purpose of this article is to derive the critical exponents, in the case of radially symmetric positive solutions, for a singular extremal operator. More precisely, we will study positive solutions of the nonlinear elliptic equation

$$|\nabla u|^{\beta}\mathcal{M}^{+}{}_{\lambda,\Lambda}(D^2u) + u^p = 0 \quad \text{in} \quad \mathbb{R}^N, \tag{1.8}$$

and extend the results of [15] for the case of $\beta \in (-1, 0)$ and obtain critical exponents. In particular, we examine the Sobolev-type exponent for (1.8).

Before we state our main theorem, we present some definitions:

Definition 1.1 Assume that u is a radial classical positive solution of equation (1.8) but in $\mathbb{R}^N\backslash\{0\}$. Then, we have

- u is a fast decaying solution if there exists $C > 0$ such that

$$\lim_{r\to\infty} r^{\tilde{N}-2}u(r) = C.$$

- u is a pseudo-slow decaying solution if there exist constants $0 < C_1 < C_2$ such that

$$C_1 = \liminf_{r\to\infty} r^{\alpha}u(r) < \limsup_{r\to\infty} r^{\alpha}u(r) = C_2,$$

  where

$$\alpha = \frac{\beta+2}{p-(\beta+1)}. \tag{1.9}$$

- u is a slow decaying solution if there exists $c_s > 0$ such that

$$\lim_{r\to\infty} r^{\alpha}u(r) = c_s.$$

- u is singular if

$$\lim_{r\to 0} u(r) = \infty$$

We state now our principal theorem.

Theorem 1.1 *Suppose $\tilde{N} > 2$ and $-1 < \beta < 0$. Then, there exists a critical exponent p_+^* such that $\max\{\tilde{p}^b, p_\beta\} < p_+^* < \tilde{p}_\beta$, where:*

$$\tilde{p}^b = \frac{1 + (\beta+1)(\tilde{N}-1)}{\tilde{N}-2},$$
$$p_\beta = \frac{(\beta+2)^2 + (\beta+1)^2(N-2)}{(\beta+1)(N-2)},$$
$$\tilde{p}_\beta = \frac{(\beta+2)^2 + (\beta+1)^2(\tilde{N}-2)}{(\beta+1)(\tilde{N}-2)},$$

such that:

1. *If $1 < p < {p^*}_+$, then there is no nontrivial radial positive viscosity solution to (1.8).*
2. *If $p = {p^*}_+$, then there is a unique, up to scaling, fast decaying radial viscosity solution to (1.8).*
3. *If ${p^*}_+ < p \le \tilde{p}_\beta$, then there is a unique, up to scaling, pseudo-slow decaying radial viscosity solution to (1.8).*
4. *If $\tilde{p}_\beta < p$, then there is a unique, up to scaling, slow decaying radial viscosity solution to (1.8).*

Remark 1.1 1. Here, solution is C^2 radial functions that satisfies Eq. (1.8) in the viscosity sense and they also satisfy equation in $\mathbb{R}^N \backslash \{0\}$ in the classical sense. For more details on the definition of viscosity solution in this setting see [11].

The next theorem exhibits the complex structure of the singular solution set for some range of p.

Theorem 1.2 *We assume that $\tilde{N} > 2$ and $p_+^* < p < \tilde{p}_\beta$, where p_+^* is given in Theorem 1.1. Then, we have the following:*

Equation (1.8) possesses at least three singular solutions u_i, $i = 1, 2, 3$, such that $u_1(r) = c_s r^{-\alpha}$,

$$c_{12} = \liminf_{r\to 0,\infty} r^\alpha u_2(r) < \limsup_{r\to 0,\infty} r^\alpha u_2(r) = c_{22},$$
$$\lim_{r\to 0} r^\alpha u_3(r) = c_s \quad \text{and} \quad c_{13} = \liminf_{r\to\infty} r^\alpha u_3(r) < \limsup_{r\to\infty} r^\alpha u_3(r) = c_{23},$$

for c_s, c_{1i} and c_{2i}, $i = 1, 2, 3$, positive constants.

Moreover, if $p = \tilde{p}_\beta$, there is a family of solutions u_μ such that

$$c_{1,\mu} = \liminf_{r\to 0,\infty} r^\alpha u_\mu(r) < \limsup_{r\to 0,\infty} r^\alpha u_\mu(r) = c_{2,\mu},$$

where $c_{1,\mu}$ and $c_{2,\mu}$ are monotonic continuous functions in $\mu \in [0, 1]$ and $c_{1,\mu} \to c_s$ and $c_{2,\mu} \to c_s$ as $\mu \to 1$ with $c_s \in I\!R^+$, recall that c_s is the constant of u_1 .

The paper is organized as follows. In Sect. 2, we present some preliminary properties of the radial solutions. In Sect. 3, we transform our problem into two systems of ordinary differential equations. By using ideas from dynamical systems, we present qualitative properties of the solutions. In the Sect. 4, we use the method introduced by Kolodner and Coffman (see [17] and [18]). This method differentiates the solution with respect to the initial value. Here, we differentiate the equation with respect to p. Then, we prove our main theorems.

2 Preliminaries

We begin by considering the initial value problem

$$u''|u'|^{\beta} = M[-\frac{(N-1)}{r}|u'|^{\beta}(m(u')) - u^p] \quad \text{in} \quad (0,+\infty) \tag{2.1}$$
$$u(0) = \gamma_0, \qquad u'(0) = 0, \tag{2.2}$$

where $\gamma_0 > 0$ and

$$M(s) = \begin{cases} \frac{s}{\Lambda} & \text{if} \quad s \geq 0, \\ \frac{s}{\lambda} & \text{if} \quad s < 0. \end{cases} \tag{2.3}$$
$$m(s) = \begin{cases} \Lambda s & \text{if} \quad s \geq 0, \\ \lambda s & \text{if} \quad s < 0. \end{cases} \tag{2.4}$$

We note that solutions of (2.1)–(2.2) are radial symmetric positive viscosity solutions of (1.8) (for details, see Lemma 3.1 in [10] and [11] for the definition of viscosity solution in this context). The existence of solution to (2.1)–(2.2) can be established using a direct fix point argument; here, we use local existence for a simpler related problem that will give existence for (2.1)–(2.2). Then, we use stable–unstable manifold for a related dynamical system that gives uniqueness for (2.1)–(2.2), see next lemma and the beginning of next section.

Remark 2.1 Assume that $u(r, p)$ is a solution of (2.1) with $u(0) = 1$ and $u'(0) = 0$, then by scaling

$$u_{\gamma_0}(r, p) = \gamma_0 u(\gamma_0^{\frac{1}{\alpha}} r, p) \quad \text{is a solution for all } \gamma_0 > 0.$$

Lemma 2.1 *There exists a solutions of*

$$\lambda|u'|^{\beta}u'' + \frac{\lambda(N-1)}{r}|u'|^{\beta}u' + u^p = 0 \quad in \quad (0, R) \tag{2.5}$$
$$u(0) = \gamma_0, \qquad u'(0) = 0 \tag{2.6}$$

that is initially decreasing and concave. Moreover, $u''(0) = 0$ and u is C^2 near zero.

Proof The existence results can be established by a fix point argument as for example in [19].

Define now $v = |u'|^{\beta}u'$, then $v(0) = 0$ since $\beta + 1 > 0$ and v also satisfies the equation

$$\lambda \frac{v'}{\beta+1} + \frac{\lambda(N-1)v}{r} = -u^p.$$

Then, taking the limit, we deduce that:

$$v'(0) = \frac{-\gamma_0^p}{\lambda(1/(\beta+1) + N - 1)} < 0.$$

Therefore, v is initially negative and decreasing, additionally since $\beta \in (-1, 0)$ we find $u''(0) = 0$ and u is C^2 near zero. □

The next lemma provides some properties of a point r_0 such that $u''(r_0) = 0$. Without losing generality, we assume now and for the rest of the paper that $\lambda = 1$.

Lemma 2.2 *Define $H(r) = \frac{N-1}{r}|u'|^{\beta}u' + u^p$. If there exists $r_0 > 0$ such that $H(r_0) = 0$ and $H'(r_0) = 0$, then $H''(r_0) < 0$. In other words H change sign with $H' \neq 0$.*

Proof If $H(r_0) = 0$ and $H'(r_0) = 0$, $H''(r_0, p) < 0$. In fact,

$$H''(r_0) = \frac{2(N-1)|u'|^\beta u'}{r_0^3} + p(p-1)u^{p-2}(u')^2 = \frac{1}{(p-1)}\frac{-u(r_0)^p}{r_0^2} < 0.$$

□

Note that this lemma is equivalent to $u''(r_0) = 0$ and u'' changing sign implies $u'''(r_0) \neq 0$.

We classify the exponent p according to the behavior of the solution of the initial value problem (2.1)–(2.2). We define:

- $\mathcal{C} = \{p \mid p > 1, u(r, p, \gamma_0)$ has a finite zero$\}$.
- $\mathcal{P} = \{p \mid p > 1, u(r, p, \gamma_0)$ is positive and pseudo-slow decaying$\}$.
- $\mathcal{S} = \{p \mid p > 1, u(r, p, \gamma_0)$ is positive and slow decaying$\}$.
- $\mathfrak{F} = \{p \mid p > 1, u(r, p, \gamma_0)$ is positive and fast decaying$\}$.

Note that these sets do not depend on the particular value of $\gamma_0 > 0$, by Remark 2.1.

3 Dynamical systems

In this section, we first use the following Emden–Fowler-type transformation introduced in [20] for the p-Laplacian. Let u be a solution of the initial value problem (2.1)–(2.2). We define

$$x(t) = u(e^t)e^{\alpha t} \quad \text{and} \quad y(t) = |u'|^\beta u'(e^t)e^{\gamma t} \quad \text{with} \quad \gamma = (\alpha+1)(\beta+1),$$

and α is define in (1.9).

Because $-1 < \beta < 0$ and M, m are Lipschitz functions from equation (2.1) we find that x, y satisfy

$$\dot{x} = \alpha x + |y|^{\frac{-\beta}{\beta+1}} y \quad \text{and} \quad \dot{y} = \tilde{\delta} y + \tilde{g}(x, y) \tag{3.1}$$

for $\tilde{\delta} = -(\beta+1)[\tilde{N} - 2 - \alpha]$ and

$$\tilde{g}(x, y) = -\tilde{\delta} y + (\beta+1)[(1+\alpha)y + M(-(N-1)m(y) - x^p)],$$

which has vanishing Lipschitz constants, as given in Theorem 4.1 (on page 330 of [21]). Therefore, because $\tilde{\delta} = -(\beta+1)[\tilde{N} - 2 - \alpha] < 0$ since $p > \tilde{p}^b$ and $\alpha > 0$, the existence and uniqueness of the stable–unstable manifold of the origin O follow. In addition, by uniqueness and Lemma 2.1, the solution of (2.5) and (2.6) is near zero the unique solution of (2.1)–(2.2). So, in particular, y is negative near $-\infty$ it remains negative, while x remains positive. Thus, x, y satisfy the following:

- If $y > \frac{-x^p}{N-1}$, defining $\delta = -(\beta+1)[N - 2 - \alpha]$, we find that:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} |y|^{\frac{-\beta}{\beta+1}} y \\ -(\beta+1)x^p \end{bmatrix}. \tag{3.2}$$

- If $y \leq \frac{-x^p}{N-1}$ we have:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \tilde{\delta} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} |y|^{\frac{-\beta}{\beta+1}} y \\ -(\beta+1)\frac{x^p}{\Lambda} \end{bmatrix}. \tag{3.3}$$

Moreover, this system has a unique critical point $P = (x(p), y(p))$ formed by $f(x(p), y(p)) = 0$, $g(x(p), y(p)) = 0$ and $y(p) \leq \frac{-x(p)^p}{N-1}$, where we denote the system (3.1) as $\dot{x} = f(x, y)$ and $\dot{y} = g(x, y)$.

Remark 3.1 Note that the unstable manifold of O, which we denote by $W^u(O)$, corresponds to solutions of (2.1)–(2.2), with

$$\lim_{t\to-\infty} e^{-\alpha t}x(t) = \gamma_0 = u(0).$$

Analyzing the stable manifold, denoted by $W^s(O)$, we have that if $(x(t), y(t)) \in W^s(O)\cap W^u(O)$ that is $x(t), y(t))$ homoclinic orbit, then u the associated solution satisfies

$$\lim_{t\to\infty} e^{-\tilde{\delta}t}y(t) = -C \quad \text{hence,} \quad \lim_{r\to\infty} \frac{|u'|^\beta u'}{r^{-(\tilde{N}-1)(\beta+1)}} = -C,$$

Using L'Hospital's rule, we can conclude that

$$\lim_{r\to\infty} u(r)r^{(\tilde{N}-2)} = \tilde{C}.$$

Thus, u is a fast decaying solution.

The phase plane in the fourth quadrant is divided into two regions by the curve

$$y = \frac{-x^p}{N-1}. \tag{3.4}$$

We denote by Q^+ the regions above (3.4) and by Q^- the regions below (3.4).

To continue, we recall the Bendixson–Dulac criterion or test (see, for example, Theorems 1 and 2 in Section 3.9 of [22], in the case of the C^1 fields).

Theorem 3.1 *Let $\dot{x} = f(x, y)$ and $\dot{y} = g(x, y)$ denote a dynamical system in the plane with Lipschitz functions f and g. Suppose that there exists ρ, a Lipschitz function (Dulac function), such that*

$$G(x, y) := \frac{\partial\, \rho f(x, y)}{\partial x} + \frac{\partial\, \rho g(x, y)}{\partial y}$$

has the same sign a.e. in D. Then, neither limit cycles nor homoclinic orbits exist in D.

Remark 3.2 Note that by Rademacher's Theorem, G is well defined a.e..

Proof First, assume that (x, y) is a homoclinic orbit in D. Then, denoting by C the corresponding closed curve and by $\hat{n}$ the is the outward-pointing unit normal, we have

$$I = \oint_C (\rho f(x, y), \rho g(x, y)) \cdot \hat{n} \mathrm{d}s = 0.$$

However, if Ω is the region surrounded by C, by the divergence theorem, we find that

$$I = \iint_\Omega G dA = 0. \tag{3.5}$$

By our assumption, G has the same sign a.e. in $\Omega \subset D$, indicating a contradiction.

The case of limit cycles is analogous, and this concludes the proof. $\square$

Lemma 3.1 *If $\alpha+\delta$ and $\alpha+\tilde{\delta}$ have the same sign, then neither limit cycles nor homoclinic orbits exist. This means in term of p that if $p>\tilde{p}_\beta>p_\beta$ or $\tilde{p}_\beta>p_\beta\geq p$ neither limit cycles nor homoclinic orbits exist for* (3.2), (3.3).

Proof We applied the Bendixson–Dulac test in Q^+:

$$G(x,y):=\frac{\partial\, f(x,y)}{\partial x}+\frac{\partial\, g(x,y)}{\partial y}=\alpha+\delta,$$

and in Q^-:

$$G(x,y):=\frac{\partial\, f(x,y)}{\partial x}+\frac{\partial\, g(x,y)}{\partial y}=\alpha+\tilde{\delta}.$$

Therefore, we can apply the Bendixson–Dulac test in fourth quadrant; hence, the result if $p>\tilde{p}_\beta>p_\beta$ or $\tilde{p}_\beta>p_\beta>p$. The case $\alpha+\delta=0$, that is, $p=p_\beta<\tilde{p}_\beta$ we still can apply Bendixson–Dulac test since if limit cycles or homoclinic exist then the interior of the orbit Ω must have positive measure in the set Q^- where we have the strict sign for G and we get a contradiction with (3.5) as in the above proof of the Bendixson–Dulac test. □

Lemma 3.2 *Let $(x(t),y(t))$ be a trajectory of* (3.2) *in $W^u(O)$; then,*

$$y\leq-\frac{(\beta+1)x^p}{1+(N-1)(\beta+1)}.$$

Proof As above, a solution to the initial value problem (2.1), (2.2) with $u''<0$, corresponds to a solution $(x(t),y(t))$ of (3.2) in $W^u(O)$. Therefore, u satisfies

$$\{|u'|^\beta u' r^{(N-1)\beta+1}\}'=-(\beta+1)u^p r^{(N-1)(\beta+1)}\quad in\quad (0,r_0),$$

where r_0 is the first point such that $u''(r_0)=0$. Integrating by parts and using $u'\leq 0$, we obtain

$$|u'|^\beta u'\leq-\frac{(\beta+1)u^p r}{1+(N-1)(\beta+1)},\quad \text{for}\quad r\leq r_0.$$

This in terms of the dynamic system is the desired inequality. □

Proposition 3.1 *We have:*

1. *If $p>\tilde{p}_\beta$, then $\quad p\in\mathcal{S}$.*
2. *If $p\leq\max\{\tilde{p}^b,p_\beta\}$, then $\quad p\in\mathcal{C}$.*

Proof From Lemma 3.2, we can prove that the solution $(x(t),y(t))$ in $W^u(O)$ is bounded because a curve of the form $y\leq -Cx^p$ crosses the curve $f(x,y)=0$ only at one point. Therefore, $x'(t)<0$ for large t, and thus, the solution is bounded.

Case 1 If $p>\tilde{p}_\beta>p_\beta$, then $\alpha+\delta<0$ and $\alpha+\tilde{\delta}<0$ thus using Lemma 3.1 we have that $(x(t),y(t))$ is not a homoclinic orbit, and it does not converge to a limit cycle. Assume now that x vanishes for some t_0. We now denote as $\mathbf{D}$ the closed curve containing the origin composed of part of the stable manifold $W^s(O)$ emanating from the origin to the second crossing with $x'=f(x,y)=0$ and we close the curve with the curve $f(x,y)=0$, all this if the second crossing point exists. Because $g(x,y)>0$ if $f(x,y)=0$, we find

$$I=\oint_{\mathbf{D}}(f(x,y),g(x,y))\cdot\hat{n}\mathrm{d}s>0.$$

Now, with $\alpha+\delta<0$ and $\alpha+\tilde{\delta}<0$, we obtain a contradiction with the divergence theorem.

If the stable manifold $W^s(O)$ has only one crossing point with $x' = f(x, y) = 0$, then this point must be P, and the same argument can be apply. Therefore, x does not vanish and does not decay fast or pseudo-slow. So, $(x(t), y(t))$ has to approach our critical point P as a consequence of Poincare–Bendixson's Theorem. Therefore, $p \in \mathcal{S}$.

Case 2 If $p \leq \tilde{p}^b$, then there is no solution, from the results of [11]. If $p \leq p_\beta$, then $\alpha+\delta \leq 0$ and $\alpha + \tilde{\delta} < 0$ using Lemma 3.1 we find that $(x(t), y(t))$ is not a homoclinic orbit, and it does not converge to a limit cycle. If we assume by contradiction that x does not vanish, then $W^u(O)$ will be connected with P. Then, we can use similar argument as above replacing $W^s(O)$ by $W^u(O)$ and obtain a contradiction. Therefore, $p \in \mathcal{C}$. □

Now, we transform the problem using the classical Emden–Fowler transformation and find that if $\tilde{H}(x, x') := (N-1)(x' - \alpha x) + \frac{x^p}{|x' - \alpha x|^\beta} \leq 0$, we get

$$
\begin{aligned}
\dot{x} &= \varrho \quad =: \quad h(x, \varrho)\\
\dot{\varrho} &= -\tilde{a}\varrho + \tilde{b}x - \frac{x^p}{\Lambda|\varrho - \alpha x|^\beta} \quad =: \quad k(x, \varrho),
\end{aligned}
\tag{3.6}
$$

$$
\text{with} \quad \tilde{a} = (\tilde{N} - 2 - 2\alpha) \quad \tilde{b} = \alpha(\tilde{N} - 2 - \alpha).
$$

We note that if $p > \tilde{p}^b$, then this system has a unique critical point that we call $T = (z(p), 0)$, characterized by $k(z(p), 0) = 0$.

Lemma 3.3 *If $\tilde{p}^b < p < \tilde{p}_\beta$, then p is not an element of the set $\mathcal{S}$ because the point T is unstable. If $p = \tilde{p}_\beta$, then the system is a center around T, that is, all trajectories close to T are periodic.*

Proof We will use the Bendixson–Dulac test with the Dulac function $\rho = |\varrho - \alpha x|^\beta$. A direct computation gives

$$
G(x, \varrho) := \frac{\partial(\rho\, h(x, \varrho))}{\partial x} + \frac{\partial(\rho\, k(x, \varrho))}{\partial \varrho} = (\alpha + \tilde{\delta})|\varrho - \alpha x|^\beta.
$$

Using $\alpha + \tilde{\delta} > 0$, because $p < \tilde{p}_\beta$, we can prove that the point T is unstable applying the same argument as in proof of Theorem 6.3 in [16], but replacing G in the divergence theorem.

We now assume that $p = \tilde{p}_\beta$. We take the orbit of (3.6) with the initial point (x_0, ϱ_0) that satisfies $\varrho_0 = (\alpha - 1/p)x_0)$ and $\tilde{H}(x_0, \rho_0) = 0$. We claim that this orbit is periodic.

In fact, if the orbit crosses the curve $\tilde{H}(x, \varrho) = 0$ again, we define the closed curve $\mathcal{D}$ composed by this orbit and closed with part of the curve $\tilde{H}$ denote by $\mathcal{C}$. Notice that on $\mathcal{C}$ we have $\varrho > (\alpha - 1/p)x$, by the definition of (x_0, ϱ_0). We will prove that

$$
(h(x, \varrho), k(x, \varrho)) \cdot \hat{n} > 0 \qquad \text{in} \quad \mathcal{C},
\tag{3.7}
$$

where n denotes the outward unite pointing normal to $\mathcal{C}$. If we compute condition (3.7) in terms of $z := \frac{\varrho}{x}$ we find

$$
w(z) := pz^2 - [\beta + 2 + (1 + 2\alpha)(\beta + 1)]z + (\beta + 1)\alpha(\alpha + 1) < 0,
$$

and therefore get $z \in [\alpha - 1/p, \alpha]$. From here, we can deduce condition (3.7) holds since $\varrho > (\alpha - 1/p)x$ on $\mathcal{C}$.

We now define Ω as the region bounded by the curve $\mathcal{D}$. Then,

$$I=\iint_{\Omega} G dA=0.$$

From the divergence theorem and (3.7), we find that I is positive, which is a contradiction.

If the solution does not cross $\tilde{H}(x,\rho)=0$ and is not periodic, then there are two points in the trajectory such that $\rho=0$. We now define a close curve composed by the orbit and closed with part of $\rho=0$. Using the divergence theorem again, we obtain a contradiction. Therefore, the claim follows.

Finally, using the last argument all orbits inside the periodic orbit find above are also periodic; these mean that the system is center around T. □

4 Sturm–Liouville identities and Coffman and Kolodner method

In this section, we study the solution near a fast decaying solution. We vary p in order to classify the solution. First, we differentiate the solution of (2.1)–(2.2) with respect to p as in [15], keeping the initial condition fixed. We prove the following two propositions that are crucial for the proof of our main results.

Proposition 4.1 *If $p^* \in \mathfrak{F}$, then for $p<p^*$ close to p^*, we have $p \in \mathcal{C}$.*

Proposition 4.2 *If $p^* \in \mathfrak{F}$, then for $p>p^*$ close to p^*, we have $p \in \mathcal{S} \cup \mathcal{P}$.*

For the proof of these propositions, we require some preliminary lemmas. Because γ_0 is kept fixed in our analysis, we do not explicitly mention it.

Let $p^* \in \mathfrak{F}$ and $u(r,p^*)$ be a solution of (2.1)–(2.2). We note that by Proposition 3.1, $p^* > \frac{1+(\beta+1)(\tilde{N}-1)}{\tilde{N}-2} = \tilde{p}^b$, and u changes its concavity only once because the solution corresponds to homoclinic orbits in the dynamical system of the previous section. Thus, there exists a unique point $r_0=r_0(p^*)$ such that $u''(r_0)=0$.

Our first step is to study the differentiation of u with respect to p, for which we need the following lemma:

Lemma 4.1 *There exist $\delta>0$ and a Lipschitz continuous function $r_0:[p^*-\delta,p^*+\delta]\to I\!R$ such that $u''(r_0(p),p)=0$.*

Proof For the proof, we will use a version of the implicit function theorem described by Goursat, see [23] that does not require differentiability in one of the variables. We define, as before, $H(r,p)=\frac{N-1}{r}|u'|^{\beta}u'+u^p$ with $H'(r_0,p^*)\neq 0$ by Lemma 2.2. Then, $L_p(r)=r-\frac{H(r,p)}{H'(r_0,p^*)}$ is a continuous and a uniform contraction in a neighborhood of r_0 to itself for all $p\in[p^*-\delta,p^*+\delta]$ and small $\delta>0$. By the Banach fixed point theorem, there exists $r_0(p)$ such that $H(r_0(p),p)=-u''(r_0(p),p)=0$. In addition, the continuity of H implies the continuity of $r_0(\cdot)$ by the triangle inequality.

We take $p=p^*$ and h small; then,

$$\begin{aligned}0=&\frac{u''\left(r_0(p+h),p\right)-u''\left(r_0(p),p\right)}{r_0(p+h)-r_0(p)}\frac{r_0(p+h)-r_0(p)}{h}\\&-\frac{u''\left(r_0(p+h),p+h\right)-u''\left(r_0(p+h),p\right)}{h}.\end{aligned}$$

Note that the last term is bounded because from (2.1), since u'' is Lipschitz when in the variables when $u'(r_0)$. Thus

$$\lim_{h\to 0}\frac{u''\left(r_0(p+h),p\right)-u''\left(r_0(p),p\right)}{r_0(p+h)-r_0(p)}=u'''\left(r_0(p),p\right)\neq 0,$$

by Lemma 2.1. Therefore, we find that r_0 is Lipschitz near p^*. □

Proposition 4.3 *We define* $\varphi(r,p)=\frac{\partial u(r,p)}{\partial p}$*; then,* $\varphi(0)=0$ *and* $\varphi'(0)=0$*, and* $\varphi(.,p^*)$ *satisfies the following equations:*

$$|u'|^\beta\left[\varphi''+\frac{(N-1)(\beta+1)u'+\beta r u''}{ru'}\varphi'\right]=-u^p\ln u-pu^{p-1}\varphi \quad \textit{if} \quad r<r_0 \qquad (4.1)$$

and

$$|u'|^\beta\left[\varphi''+\frac{(\tilde{N}-1)(\beta+1)u'+\beta r u''}{ru'}\varphi'\right]=-\frac{1}{\Lambda}u^p\ln u-pu^{p-1}\varphi \quad \textit{if} \quad r>r_0. \qquad (4.2)$$

Proof Using Gronwall's inequality, Arzela–Ascoli's Theorem and Lebesgue's Dominated Convergence Theorem, the differentiability follows, using similar arguments as those used for the example in [24] for r near zero, since Pucci's operator is the Laplacian by the fact that $u''(r,p)<0$ for small r.

Therefore, we have found the solution φ to (4.1) in $(0,r_1]$ with $r_1<r_0(p^*)$. This solution can be continuously extended (using a local fixed point argument) to a solution of (4.1) and (4.2) in all $(0,+\infty)$ by

$$\left(|u^*|^\beta\right)\varphi'=-\int_{r_1}^{r}\rho_u(s)\frac{\left(p^*u^{p^*-1}\phi+u^{p^*}\log u\right)}{\theta_u}+\left(|u'|^\beta\right)\varphi'(r_1).$$

Here, we use the notation $u:=u(p^*)$, $N_u=(N-1)/\theta_u$, where $\theta_u(r)=1$ for $r\le r_0(p^*)$, $\theta_u(r)=\Lambda$ for $r>r_0(p^*)$, and

$$\rho_u(r)=e^{-(\beta+1)\int_{\bar r}^{r}e^{\frac{N_u(s)-1}{s}}\mathrm{d}s}.$$

We now define $v:=u(p^*+h)$ and $w_h=(u-v)/h$. Using the equations for u and v, we find that

$$\begin{aligned}&|u'|^\beta\left(\theta_u w_h''+\frac{(N-1)w_h'}{r}\right)+\left(|u'|^\beta-|v'|^\beta\right)\left(\theta_u v''+\frac{(N-1)v'}{r}\right)+|v'|^\beta)\\&+(\theta_u-\theta_v)|v'|^\beta v''+u^p-v^p=0.\end{aligned}$$

Here, θ_v is analogous to θ_u, depending on the sign of v''.

We now note that

$$|E(h)|:=|\frac{1}{h}\int_{r_0(p^*+h)}^{r_0(p^*)}\rho_u(s)(\theta_u-\theta_v)|v'|^\beta v''\mathrm{d}s|\le c|v''(\xi)|,$$

where we use the fact that r_0 is Lipschitz and $\xi\in(r_0(p^*+h),r_0(p^*))$ and assume without loss of generality that $r_0(p^*+h)<r_0(p^*)$. Thus, $E(h)\to 0$ as $h\to 0$.

Let us now define $\phi(r)=w_h-\varphi$. Using the Mean Value Theorem, we then find that

$$(|u'|^\beta)\phi'=-\int_{r_1}^{r}\rho_u(s)\left(\frac{(p^*u^{p^*-1}\phi)}{\theta_u}+o(h)\right)\mathrm{d}s,$$

with $o(h) \to 0$ as $h \to 0$, by the above bounds. We now use Gronwall's inequality to conclude that $\phi \to 0$ and $\phi' \to 0$ as $h \to 0$ on $[r_1, \infty)$ because $\phi(r_1) \to 0$ and $\phi'(r_1) \to 0$ as $h \to 0$. We obtain the differentiability of u with respect to p and the equations for φ. $\square$

We now obtain the following Sturm–Liouville identities satisfied by $u = u(p^*)$ and φ.

Lemma 4.2 *Let u and φ be given as above. Then, we have the following identities:*

$$\left\{\rho_u |u'|^\beta [(u')\varphi - u(\varphi)']\right\}' = \frac{\rho_u}{\theta_u}\left[[p - (\beta+1)]\, u^p \varphi + u^{p+1} \ln u\right], \tag{4.3}$$

$$\left\{\rho_{p^*} |u'|^\beta [(ru)''\varphi - (ru)'(\varphi)']\right\}' = \frac{\rho_u}{\theta_u}\left[[p - (2\beta+3)]\, u^p \varphi + u^{p+1} \ln u + r u^p \ln u u'\right]. \tag{4.4}$$

Proof These identities follow using the equations satisfied by φ and u. $\square$

Lemma 4.3 *It is not possible to satisfy the following simultaneously:*

$$\lim_{r\to\infty} \varphi(r) = c_1 \le 0 \quad \text{and} \quad \lim_{r\to\infty} r\varphi'(r) = 0.$$

Proof Because u is a fast decaying solution, there exists $C > 0$ such that

$$\lim_{r\to\infty} u(r) r^{\tilde N - 2} = C \quad \text{and} \quad \lim_{r\to\infty} u'(r) r^{\tilde N - 1} = (2 - \tilde N)C. \tag{4.5}$$

Then,

$$\lim_{r\to\infty} |u'(r)|^\beta u'(r) r^{(\tilde N - 1)(\beta+1)} = C^{\beta+1}(2 - \tilde N)(\tilde N - 2)^\beta. \tag{4.6}$$

We now use our assumption on φ to find:

1. $\lim_{r\to\infty} r^{(\tilde N - 1)(\beta+1)} |u'(r)|^\beta [u'\varphi - u\varphi'] = C^{\beta+1}(\tilde N - 2)^\beta (2 - \tilde N) c_1$.
2. $\lim_{r\to\infty} r^{(\tilde N - 1)(\beta+1)} |u'(r)|^\beta [(ru)''\varphi - (ru)'\varphi'] = C^{\beta+1}(\tilde N - 2)^\beta (2 - \tilde N)(3 - \tilde N) c_1$.

For the second limit, we use $\lim_{r\to\infty} r u^p(r)\varphi r^{(\tilde N - 1)(\beta+1)} = 0$, as u is fast decaying and $p > \tilde p^b$, which yields $-[(\beta+1)(\tilde N - 1) + 1] - p(\tilde N - 2) < 0$.

We now integrate the identities (4.3) and (4.4) to obtain

$$\int_0^\infty \frac{\rho_u}{\theta_u}\left[[p - (2\beta+3)]\, u^p\varphi + u^{p+1}\ln u + r u^p \ln u u'\right]$$
$$= C^{\beta+1}(\tilde N - 2)^\beta (2 - \tilde N)(3 - \tilde N) c_1,$$

and

$$\int_0^\infty \frac{\rho_u}{\theta_u}\left[[p - (\beta+1)]\, u^p\varphi + u^{p+1}\ln u\right] = C^{\beta+1}(\tilde N - 2)^\beta (2 - \tilde N) c_1.$$

If we multiply the first integral by $\frac{p-(2\beta+3)}{p-(\beta+1)}$ and subtract the second one, we obtain

$$\int_0^\infty \frac{\rho_u}{\theta_u}\left[(\alpha u + r u') u^p \ln u\right] = \left[3 - \tilde N - \frac{p - (2\beta+3)}{p - (\beta+1)}\right] C^{\beta+1}(\tilde N - 2)^\beta (2 - \tilde N) c_1.$$

We observe that in the Emden–Fowler variables $x'(t) = r^\alpha(\alpha u + r u')$, we can choose γ (choose T for $x(\cdot + T)$) such that $\alpha u + r u'$ changes sign when u is 1. We now have $(\alpha u + r u') u^p \ln u > 0$, $\forall r \ge 0$. Using $p > \tilde p^b$, we find that the right-hand side is not positive; thus, we obtain a contradiction. $\square$

Continuing with our analysis, we define the function

$$w = w_\theta(r) = r^\theta u(r, p).$$

Then, w satisfies the equation:

$$w'' + \frac{\left(\tilde{N} - 1 - 2\theta\right)}{r} w' + \frac{\theta\left(\theta - \tilde{N} + 2\right)}{r^2} w = -\frac{r^\theta u^p}{\Lambda |u'|^\beta} \quad \text{for} \quad r > r_0. \tag{4.7}$$

We now choose $\theta > 0$ as $\theta = \frac{(\tilde{N}-1)}{2}$ if $\tilde{N} > 3$ and $\theta = \frac{(\tilde{N}-2)}{2}$ if $2 < \tilde{N} \leq 3$. This function was introduced by Erbe and Tang [25] for a related problem.

In what follows, we assume that $\tilde{N} > 3$. The case $2 < \tilde{N} \leq 3$ can be addressed using similar arguments.

We define the following function:

$$y(r) = \frac{\partial w(r)}{\partial p} = r^\theta \varphi;$$

then, y satisfies the equation

$$\begin{aligned} &|u'|^\beta \Bigg[y'' - \beta \frac{1}{\Lambda} u^p (u')^{-1} |u'|^{-\beta} y' + \Bigg(\frac{(\tilde{N}-1)(3-\tilde{N})}{4r^2} + \frac{\beta u^p (u')^{-1} |u'|^{-\beta} (\tilde{N}-1)}{2\Lambda r} \\ &\quad + \frac{1}{\Lambda} p u^{p-1} |u'|^{-\beta} \Bigg) y \Bigg] \\ &= -\frac{1}{\Lambda} r^{-\theta} u^p \ln u \quad \text{for} \quad r > r_0. \end{aligned} \tag{4.8}$$

Observe that

$$C_1(r) := \left[\frac{(\tilde{N}-1)(3-\tilde{N})}{4r^2} + \frac{\beta u^p (u')^{-1} |u'|^{-\beta} (\tilde{N}-1)}{2r} + \frac{1}{\Lambda} p u^{p-1} |u'|^{-\beta} \right] < 0, \tag{4.9}$$

for large r, as u is a fast decaying solution, $p > \tilde{p}^b$ and $\beta < 0$.

We now prove the following lemma that describes the asymptotic behavior of y.

Lemma 4.4 *The function y defined above satisfies $y(r) > 0$ for large r.*

Proof Suppose by contradiction that there exists a large $\bar{r}$ such that $y(\bar{r}) \leq 0$. We then have the following two possibilities:

(a) $y(r) \leq 0$, for all $r \geq \bar{r}$ or
(b) there exists $r^* > \bar{r}$, and there exists $y(r^*) > 0$.

- First, we prove part (a):
  We have $\varphi(r) \leq 0$ for all $r \geq \bar{r}$, and from (4.3), we have for large r,

$$\begin{aligned} &\left\{ r^{(\tilde{N}-1)(\beta+1)} |u'|^\beta \left[(u')\varphi - u(\varphi)' \right] \right\}' \\ &\quad = r^{(\tilde{N}-1)(\beta+1)} \left[[p - (\beta+1)] u^p \varphi + u^{p+1} \ln u \right] < 0 \\ &\text{and} \\ &\left\{ r^{(\tilde{N}-1)(\beta+1)} |u'|^\beta (\varphi)' \right\}' = -\left[u^p \ln u + p u^{p-1} \varphi \right] > 0. \end{aligned} \tag{4.10}$$

  Again, there exist two possibilities:

1. There exists $r^* \geq \bar{r}$ such that $|u'|^\beta(u'(r^*)\varphi(r^*) - u(r^*)\varphi'(r^*)) \leq 0$ or
2. for all $r \geq \bar{r}$, $|u'|^\beta(u'(r)\varphi(r) - u(r)\varphi'(r)) > 0$.

Case 1 From (4.10), we have $|u'|^\beta(u'(r)\varphi(r) - u(r)\varphi'(r)) < 0$ for all $r \geq r^*$, by which it follows that the function $\frac{u}{\varphi}$ is decreasing. Thus, there is a number c_∞, possibly $-\infty$, such that

$$\lim_{r\to\infty} \frac{u(r)|u'|^\beta r^{(\tilde{N}-1)(\beta+1)-1}}{\varphi(r)|u'|^\beta r^{(\tilde{N}-1)(\beta+1)-1}} = c_\infty.$$

We use that u is fast decaying to find

$$\lim_{r\to\infty} \varphi(r)|u'|^\beta r^{(\tilde{N}-1)(\beta+1)-1} = \frac{C^{\beta+1}(\tilde{N}-2)^\beta}{c_\infty} \leq 0. \tag{4.11}$$

Because $\{r^{(\tilde{N}-1)(\beta+1)}|u'|^\beta(\varphi)'\}' > 0$ for large r, there is a positive constant c_1, possibly $+\infty$, such that

$$\lim_{r\to\infty} \varphi'(r)|u'|^\beta r^{(\tilde{N}-1)(\beta+1)} = c_1.$$

Thus,

$$\lim_{r\to\infty} \varphi'(r) r^{(\tilde{N}-1)} = C_1,$$

for some C_1, possibly $+\infty$. Hence, by *L'Hospital's rule*, we obtain

$$\lim_{r\to\infty} \varphi'(r) r^{(\tilde{N}-1)} = (2-\tilde{N}) \lim_{r\to\infty} \varphi(r) r^{(\tilde{N}-2)} < +\infty.$$

We find that $\varphi(r) \longrightarrow 0$ and $r\varphi'(r) \longrightarrow 0$ as $r \longrightarrow \infty$, contradicting Lemma 4.3 .

Case 2 From (4.10), there exists $c_2 \in (-\infty, \infty]$ such that $\lim_{r\to\infty} \varphi'(r)|u'|^\beta r^{(N-1)(\beta+1)} = c_2$.
In the case that $c_2 \leq 0$, we have $\varphi'(r) < 0$ for all large r; consequently, there exists $c_1 \in [-\infty, 0)$ such that $\lim_{r\to\infty} \varphi(r) = c_1$. We claim that c_1 is finite. In fact, we first observe that because $\varphi'(r)|u'|^\beta r^{(\tilde{N}-1)(\beta+1)} = r^{(\tilde{N}-1)(\beta+1)-1}|u'|^\beta(r\varphi'(r))$ converge to a finite limit, we necessarily have that $\lim_{r\to\infty}(r\varphi'(r)) = 0$.
Then, from (4.10) and the assumption used for case 2, we find a finite constant $c \geq 0$ such that

$$\lim_{r\to\infty} r^{(\tilde{N}-1)(\beta+1)}|u'|^\beta \left(u'(r)\varphi(r) - u(r)\varphi'(r)\right) = c. \tag{4.12}$$

It follows that c_1 is finite; thus, we obtain a contradiction with Lemma 4.3.
In the case that $c_2 > 0$, $\varphi'(r) > 0$ for all large r such that there exists a constant $c_1 \in (-\infty, 0]$ and $\lim_{r\to\infty} \varphi(r) = c_1$.
We have (4.12), from where we find a nonnegative constant c_3 such that $\lim_{r\to\infty} r\varphi'(r) = c_3$.
If $c_3 > 0$, then integrating this last limit, we conclude that φ is unbounded, which is impossible. Thus, $c_3 = 0$, and we again obtain a contradiction using Lemma 4.3.

- Now, we prove part (b):
If there exists $r_1 > r^*$, a first point such that $y(r_1) = 0$, then there exists a local maximum of y for some r_2 with $y(r_2) > 0$. However, this contradicts equation (4.8), which implies $y''(r_2) > 0$. □

Lemma 4.5 *The function y defined above satisfies $y'(r) > 0$ for large r.*

Proof By (4.8), we have that $y(r)$ is strictly monotone for large r. Thus,

$$\lim_{r\to+\infty} y(r) = L \in [0, +\infty].$$

If $L < +\infty$, we obtain $\lim_{r\to+\infty} y'(r) = 0$ that is in contradiction with Lemma 4.3, and if $L = +\infty$, we necessarily have $y'(r) > 0$ for large r. □

We are now ready to prove Proposition 4.1.

Proof of Proposition 4.1 We return to the notation $p^* \in \mathfrak{F}$. If $p^* \in \mathfrak{F}$ and $p < p^*$ is close enough to p^*, we suppose that $u(0, p) = u(0, p^*) = \gamma$, where γ was defined in Lemma 4.3.

We suppose that $p \in \mathfrak{F}$ and, as above, $\tilde{N} > 3$. We also define

$$w(r) = r^{\frac{\tilde{N}-1}{2}} u(r, p), \qquad w_*(r) = r^{\frac{\tilde{N}-1}{2}} u(r, p^*)$$

and $v = w_* - w$. We have $w(r)$ satisfied by the definition of the Pucci operator

$$|u'(r, p)|^\beta \left(w''(r) + \frac{(\tilde{N}-1)(3-\tilde{N})}{4r^2} w(r) \right) \le -r^\theta \frac{u(r, p)^p}{\Lambda}.$$

The equation for $w_*(r)$ is

$$|u'(r, p^*)|^\beta \left(w_*''(r) + \frac{(\tilde{N}-1)(3-\tilde{N})}{4r^2} w_*(r) \right) = -r^\theta \frac{u(r, p^*)^{p^*}}{\Lambda}.$$

Then, v satisfies the equation:

$$|u'(r, p^*)|^\beta \left(v'' + \frac{(\tilde{N}-1)(3-\tilde{N})}{4r^2} v \right) + \left(|u'(r, p^*)|^\beta - |u'(r, p)|^\beta\right) \left(\frac{r^{\frac{\tilde{N}-1}{2}}}{\theta_{u(p)}} u^p |u'|^{-\beta} \right)$$
$$+ r^\theta \frac{u(r, p^*)^{p^*} - u(r, p)^p}{\Lambda} \ge 0,$$

for large r. From the Mean Value Theorem, we find $\tau(r) \in (\min\{u'(r, p), u'(r, p^*)\}, \max\{u'(r, p), u'(r, p^*)\})$ and $\kappa(r) \in (\min u(r, p), \max u(r, p^*))$ such that

$$\begin{aligned} &|u(r, p^*)'|^\beta (v'' + \frac{\beta}{\theta_{u(p)}} |\tau(r)|^{\beta-1} u^p |u'|^{-\beta} v' \\ &+ \left[\frac{(\tilde{N}-1)(3-\tilde{N})}{4r^2} + \frac{\beta |\tau(r)|^{\beta-1} u^p |u'|^{-\beta} (\tilde{N}-1)}{2\theta_{u(p)} r} \right. \\ &\left. + r^\theta \frac{p^*(\kappa(r))^{p^*-1}}{\Lambda} \right] v + r^\theta \frac{u(r, p)^{p^*} - u(r, p)^p}{\Lambda} \ge 0 \quad \text{for all} \quad r^* \ge \bar{r}. \end{aligned} \tag{4.13}$$

We now use the continuity of the solution (2.1)–(2.2) with respect to the parameter p, as well as the fact that $u'(r, p) < 0$ for all $r > 0$, to find $\bar{r}$ and $\epsilon > 0$ such that $u(r, p) < 1$ for all $r \ge \bar{r}$ and all $p \in (p^* - \epsilon, p^*)$. Thus, v satisfies the same inequality, but without the last term.

Using the same argument that we used for the equation satisfied by y, we find that if $v'(\hat{r}) = 0$ for some large $\hat{r}$, then $v''(\hat{r}) > 0$.

From Lemmas 4.4 and 4.5, there exists a $\tilde{r}$ such that $y(\tilde{r}) > 0$ and $y'(\tilde{r}) > 0$. Thus, $v(\tilde{r}) > 0$ and $v'(\tilde{r}) > 0$ for $p \in (p^* - \varepsilon, p^*)$ near p^*. However, as $p \in \mathfrak{F} \cup \mathcal{S} \cup \mathcal{P}$ and $\tilde{N} > 3$, we have $v(r) \to 0$ as $r \to \infty$ or that $v(r)$ is negative for large r. Accordingly, v has a positive maximum, which brings us to a contradiction. □

Proof of Proposition 4.2 Assume that $p \in \mathcal{C}$ and $p > p^*$. We proceed in a similar way as in the previous proposition, where, this time, v satisfies the inverse inequality. Next, we find that there exists a large $\tilde{r}$ such that $v(\tilde{r}) < 0$ and $v'(\tilde{r}) < 0$. This brings us to a contradiction with the equation satisfied by v. □

Lemma 4.6 *$\mathfrak{F}$ is a singleton.*

Proof Note that the sets $\mathcal{C}$ and $\mathcal{S} \cup \mathcal{P}$ are open. With the above propositions, we find that $\mathfrak{F}$ is a singleton.

Finally, we present the proof of our principal theorems. □

Proof of Theorem 1.1 From Lemma 4.6, $\mathfrak{F}$ is a singleton. Then, Proposition 3.1 1) implies that $\max\{\tilde{p}_+^b, p_\beta\} < p_+^* < \tilde{p}_\beta$. Proposition 4.1 implies that if $1 < p < p_+^* \in \mathfrak{F}$, there is no nontrivial radial solution to (2.1), (2.2) (all solutions are crossing). Moreover, we claim that all radial viscosity solution of (1.7) satisfies (2.1), (2.2). In fact, by the strong maximum principle (see [11]) nontrivial solution is positive. Using the maximum principle (see [11]), that $|\nabla u|^\beta \mathcal{M}^+{}_{\lambda,\Lambda}(D^2u) < 0$ in $B(0, R)$ and that constants are subsolution of $|\nabla u|^\beta \mathcal{M}^+{}_{\lambda,\Lambda}(D^2u) = 0$, we get that u is decreasing and so the origin is a local maximum.

Now we use the regularity results of [26] to get that the solution is $C^{1,\gamma}$ and therefore $u'(0) = 0$. Moreover, using Proposition 1.1 of [26], we get that if u is a solution of (1.7) then it satisfies

$$\mathcal{M}^+{}_{\lambda,\Lambda}(D^2u) = -u^p|\nabla u|^{-\beta} \quad \text{in} \quad \mathbb{R}^N,$$

from here by the convexity of $\mathcal{M}^+{}_{\lambda,\Lambda}$ we get by using Evans-type regularity results that u is C^2 (for more details see [27,28] and [8]). Therefore, u satisfies (2.1), (2.2) and the claim follows.

From Proposition 4.2, we have that $p_+^* < p \in (\mathcal{S} \cup \mathcal{P})$. Using Lemma 3.3, we obtain that if $p_+^* < p \le \tilde{p}_\beta$, then $u(p)$ is a pseudo-slow decaying solution.

By Proposition 3.1 2, if $\tilde{p}_\beta < p$, then $u(p)$ is a slow decaying solution. □

Proof of Theorem 1.2 That (1.8) has at least three singular solutions u_i, $i = 1, 2, 3$, follows by Theorem 1.1 and the analysis of the dynamical system in Lemma 3.3 and Poincare–Bendixson's Theorem. In fact, if $p_+^* < p < \tilde{p}_\beta$, then the point T in the dynamical system is unstable by Lemma 3.3 and bound by a periodic orbit found in part (iii) of Theorem 1.1 so we have the existence of the extra u_3 singular solution. In addition, if $p = \tilde{p}_\beta$, the system is a center around T and the family of periodic orbit around T give rise to the family u_μ, $\mu \in [0, 1]$. □

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