CONCENTRATION ON SUBMANIFOLDS FOR AN AMBROSETTI-PRODI TYPE PROBLEM

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Abstract

Given a smooth bounded domain Ω of $\mathbb{R}^n$ and consider the problem

$$\left\{ \begin{array}{ll} -\Delta u = |u|^p - t\,\psi & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{array} \right. \tag{0.1}$$

where t is a large positive parameter, $p > 1$ and ψ is an eigenfunction of $-\Delta$ with Dirichlet boundary condition corresponding to the first eigenvalue λ_1. Assuming that Ω contains a k-dimensional compact submanifold K which is stationary and non-degenerate for the weighted functional

$$\int_K \psi^{(1-\frac{1}{p})(\frac{p+1}{p-1}-\frac{n-k}{2})} dvol$$

such that $\mathrm{dist}(K, \partial\Omega) > \delta_0 > 0$ then for $1 < p < \frac{n+2-k}{n-2-k}$ we prove the existence of a sequence $t = t_j \to \infty$ and solutions u_t that concentrate along K. This result proves in particular the validity of a conjecture by Hollman-Mckenna in full generality, see [19], extending the result in [4] where the case $n = 2$ and $k = 1$ has been considered.

Keywords: Concentration phenomena, Infinite dimensional reduction, non-degenerate stationary submanifolds.

AMS subject classification: 35J25; 35J20; 35B33; 35B40.

Contents

1. Introduction and statement of main results

Given a smooth bounded domain Ω of $\mathbb{R}^n$, $n \geq 3$ and consider the following elliptic Ambrosetti-Prodi type problem

$$\begin{cases} -\Delta u = g(u) - t\psi & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where t is a positive real parameter, ψ is an eigenfunction of $-\Delta$ corresponding to the first Dirichlet eigenvalue λ_1, and the function g satisfies

$$\lim_{t\to+\infty} \frac{g(t)}{t} = \mu > \lambda_1 > \lim_{t\to-\infty} \frac{g(t)}{t} = \nu.$$

Here, $\mu = +\infty$ and $\nu = -\infty$ are allowed.
This problem have attracted lot of attention in the last decades and many works have been devoted in understanding the number of solutions, we refer the interested readers to [2, 3] and the references therein. It has been proved that if $g(t)$ grows sub-critically at ∞, then problem (1.1) has at least two solutions: one is a local minimum of the Euler Lagrange functional and the other is a mountain-pass type solution. If $g(\xi) = \xi^2$ and Ω is a unit square in $\mathbb{R}^2$, Bruer-McKenna-Plum [3] showed, using a computer assisted proof, that (1.1) admits at least four solutions. If Ω is the unit ball and $g(\xi) = \xi^2$ de Figueiredo-Santra-Srikanth [11] proved that (1.1) possesses a non-radially symmetric solution for $t > 0$ large. Later Dancer-Yan [7, 8], considered the case where $g(\xi) = |\xi|^p$ with p subcritical ($1 < p < \dfrac{N+2}{N-2}$, $N \geq 3$ and $p \in (1, +\infty)$ for $N = 2$). They constructed solutions with sharp peaks near local maximum points of ψ or near the boundary. Solutions with arbitrarily many peaks, as $t \to \infty$; has been also constructed. This proves in particular a conjecture due to Lazer and Mckenna, see [23]. These results have been also extended to different kind of nonlinearities, see for instance [14, 8, 9, 24, 25, 37, 35] and some references therein.

All the a results mentioned above concern point concentrating solutions. *A natural and interesting question is then whether solutions exhibiting concentration on higher dimensional sets exist.* This is in fact a conjecture formulated by Hollman-McKenna in [19] based on some numerical evidences. For solutions concentrating on higher dimensional sets for equation (1.1), we are aware of very few results. The one-dimensional case with $N = 2$ has been recently studied by Bakhti-Santra [4]. They proved that given a closed curve Γ in Ω, which is stationary and nondegenerate with respect to the weighted functional $\int_\Gamma \psi^{\frac{p+3}{2p}}$ and which satisfies $\operatorname{dist}(\Gamma, \partial\Omega) > 0$, then as ε tends to zero and satisfying some gap condition (to be described later), problem (1.1) has a solution u_ε which concentrates near Γ. The gap condition is in fact related to a resonance phenomenon, which is a typical phenomenon for concentration on positive dimensional sets and that one meets in the study of higher dimensional concentration for several problems such as singularly perturbed equations, nonlinear Shrödinger equation, constant mean curvature hypersurfaces, etc. These results shows in fact that high dimensional-bubbling phenomenon is conceptually quite different to point bubbling.
Note that setting $\varepsilon^2 = t^{-(p-1)/p}$, it is easy to check that u is a solution of (1.1) (with $g(u) = |u|^p$) if and only if $t^{-1/p}u$ is solution of

$$\begin{cases} -\varepsilon^2 \Delta u = |u|^p - \psi & \text{on } \Omega, \\ u = 0 & \text{in } \partial\Omega. \end{cases} \tag{1.2}$$

The main purpose of this paper is to prove existence of solutions to this above problem that concentrate at a submanifold K of dimension $1 \leq k \leq n-1$ with $1 < p < \frac{n-k+2}{n-k-2}$ if $n-k \geq 3$ and $p \in (1, +\infty)$ if $n-k = 1, 2$.

Noticing the similarities between the above Ambrosetti-Prodi type problem and the nonlinear Schrödinger equation with potential in the whole space or in a compact riemannian manifold without boundary. The latter one has been extensively studied in the last decades and solutions concentrating at points or high dimensional sets has been obtained. We refers the reader to del Pino-Kowalczyk-Wei [12] for curve concentration, Mahmoudi-Malchiodi-Montenegro [30] for curve concentration but for complex-valued solution with highly oscillatory phase and Mahmoudi-Sánchez-Yao [26] for higher dimensional concentration. See also [13, 10, 16, 28, 29, 31, 32, 33, 34, 38] for some related problems.

If one looks formally for solutions that concentrate near a submanifold K of dimension k, we choose an appropriate system of coordinates (Fermi coordinates) $(y, z) \in K \times \mathbb{R}^{n-k}$. Then, scaling on $n-k$ variable, the profile of solutions concentrating near K is given by the *ground state* $U_{\psi(y)}$ of the limiting equation

$$-\Delta u + \psi(y, 0) - u^p = 0 \quad \text{in} \quad \mathbb{R}^{n-k}. \tag{1.3}$$

Then one formally looks for solutions which behave qualitatively like

$$u_\varepsilon(x) \sim U_{\psi(y)}\Big(\frac{x - \Phi(y)}{\varepsilon}\Big), \quad \text{as } \varepsilon \text{ tends to zero} \tag{1.4}$$

where Φ is a normal section defined on K. Since $U_{\psi(y)}$ decays exponentially to 0 at infinity, u_ε vanishes rapidly away from K.

Note that unlike the point concentration case, the limit set is not stationary for the potential ψ. Indeed, define the energy functional

$$E(u) = \int_\Omega \Big(\frac{\varepsilon^2}{2}|\nabla_{\bar{g}} u|^2 + \psi(z)u\Big) - \frac{1}{p+1}\int_\Omega |u|^{p+1}. \tag{1.5}$$

If we look for solution of the forme $u = U_K + \phi$ where U_K is an approximate solution with leading term given by the function u_ε defined in (1.4), then performing the change of variables

$$u(x) = \psi(x_0)^\alpha v\Big(\psi(x_0)^\beta\, x\Big)$$

with an appropriate choice of the constants α and β, one gets that

$$E(U_K) \sim \varepsilon^{n-k}\int_K \psi^\sigma dvol, \qquad \text{with} \qquad \sigma = \frac{p-1}{p}\Big(\frac{p+1}{p-1} - \frac{n-k}{2}\Big).$$

Based on the above energy considerations, one can suspect that *concentration on k-dimensional sets for $k = 1, \cdots, n-1$ is expected* under suitable stationary and non-degeneracy assumptions on the limit set K.

By adapting an infinite dimensional version of the Lyapunov-Schmidt reduction method developed in [12], Manna-Santra [12] successfully proved the validity of the Hollman-Mckenna conjecture for $n = 2$ and $k = 1$. The main aim of this paper is to generalize this result to any dimension and codimension. For this purpose, we first recall the key steps in [4] [12] and [39]. The first main step is the construction of proper approximate solutions for general submanifolds, to this aim we first expand the Laplace-Betrami operator for arbitrary submanifolds, see Proposition 2.1. Then by an iterative scheme of Picard's type, a family of very accurate approximate solutions can be obtained, see Section 3. Next we develop an infinite dimensional reduction such that the construction of positive solutions of problem (1.1) can be reduced to the solvability of a reduced system (4.10). For more details about the setting-up of the problem, we refer the reader to Subsection 4.1. Our main result of this paper is the following:

Theorem 1.1. *Let Ω be a smooth n-dimensional bounded domain and let $\psi : \Omega \to \mathbb{R}$ be an eigenfunction of $-\Delta$ with Dirichlet boundary condition corresponding to the first eigenvalue λ_1. Given $k = 1, \dots, n-1$, and $1 < p < \frac{n+2-k}{n-2-k}$. Suppose that K be a stationary non-degenerate smooth compact submanifold in Ω for the weighted functional*

$$\int_K \psi^{(1-\frac{1}{p})(\frac{p+1}{p-1}-\frac{n-k}{2})} dvol$$

with $\operatorname{dist}(K, \partial\Omega) > \delta > 0$, then there is a sequence $\varepsilon_j \to 0$ such that problem (1.1) possesses solutions u_{ε_j} which concentrate near K. Moreover, for some constants C, $c_0 > 0$, the solutions u_{ε_j} satisfies globally

$$|u_{\varepsilon_j}(z)| \le C \exp\big(-c_0\, dist(z, K)/\varepsilon_j\big).$$

The gap condition on ε is due to a resonance phenomena, namely the existence of values of ε for which the linearised operator is not invertible. Similar conditions can be found in [4, 12, 39] and some references therein.

Before closing this introduction, we notice that problem (1.1) is similar to the following singular perturbation problem

$$\varepsilon^2 \Delta_{\bar{g}} u - V(z)u + u^p = 0 \quad in\ M, \tag{1.6}$$

where $(M, \bar{g})$ is a compact smooth n-dimensional Riemannian manifold without boundary or the Euclidean space $\mathbb{R}^n$, ε is a small positive parameter, $p > 1$ and V is a uniformly positive smooth potential. In [26], Mahmoudi and al. proove the following result: Given $k = 1, ..., n-1$, and $1 < p < \frac{n+2-k}{n-2-k}$. Assuming that K is a $k-$dimensional smooth, embedded compact submanifold of M, which is stationary and non-degenerate with respect to the functional $\int_K V^{\frac{p+1}{p-1}-\frac{n-k}{2}} dvol$, we prove the existence of a sequence $\varepsilon = \varepsilon_j \to 0$ and positive solutions u_ε that concentrate along K. This result proves in particular the validity of a conjecture by Ambrosetti and al. [1], extending a recent result by Wang and al. [39], where the one co-dimensional case has been considered. This latter problem arises in the study of some biological models and as (1.1) it exhibits concentration of solutions at some points of $\overline{\Omega}$. Since this equation is homogeneous, then the location of concentration points is determined by the geometry of the domain. On the other hand, it has been proven that solutions exhibiting concentration on higher dimensional sets exist. For results in this direction we refer the reader to [13, 28, 29, 31, 32, 33, 38].

The paper is organized as follows. In Section 2 we introduce the Fermi coordinates in a tubular neighborhood of K in M and we expand the Laplace-Beltrami operator in these Fermi coordinates. In Section 3, a family of very accurate approximate solutions is constructed. Section 4 will be devoted to develop an infinite dimensional Lyapunov-Schmidt reduction and to prove Theorem 1.1.

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2. Preliminary background

This section will devoted to introducing some geometric background like Fermi coordinates which play important role in the higher dimensional concentrations. We first introduce the auxiliary weighted functional corresponding to problem (1.1).

2.1. Stationary, non-degenerate submanifolds. Given a k-dimensional compact submanifold K of $\mathbb{R}^n$, $1 \le k \le n-1$ and let $\{K_t\}_t$ be a smooth one-parameter family of submanifolds such that $K_0 = K$. We define weighted functional

$$\mathcal{E}(t) = \int_{K_t} \psi^\sigma dvol, \qquad \text{with} \qquad \sigma = (1-\frac{1}{p})(\frac{p+1}{p-1} - \frac{n-k}{2}). \tag{2.1}$$

Denote ∇^T and ∇^N to be connections projected to the tangential and normal spaces on K. We give the following definitions on K which appeared in Theorem 1.1.

Definition 2.1 (Stationary condition)**.** *A submanifold K is said to be stationary relative to the functional $\int_K \psi^\sigma dvol$ if*

$$\sigma \nabla^N \psi = -\psi H \ \text{ on } K, \tag{2.2}$$

where H is the mean curvature vector on K.

Definition 2.2 (Nondegeneracy (ND) condition)**.** *We say that K is non-degenerate if the quadratic form*

$$\begin{aligned}\int_K \Big\{ \Big\langle \Delta_K \Phi + \frac{\sigma}{\psi} \nabla_K \psi \cdot \nabla_K \Phi, \Phi \Big\rangle + \frac{\sigma^{-1}}{(n-k)p} H(\Phi)^2 - \frac{\sigma}{\psi} (\nabla^N)^2 \psi\, [\Phi, \Phi] \\ + \Gamma_b^a(\Phi) \Gamma_a^b(\Phi) \Big\} \psi^\sigma \sqrt{\det(g)}\end{aligned} \tag{2.3}$$

defined on the normal bundle to K, is non-degenerate.

Here and in the rest of this paper, Einstein summation convention is used, that is, summation over repeated indices is understood.

2.2. Expansion of the metric near geodesic normal coordinates. Let K be a k-dimensional submanifold of $\mathbb{R}^n$ contained in Ω such that $\mathrm{dist(K, \partial\Omega)} > \delta_0 > 0$ $(1 \le k \le n-1)$. Define $N = n-k$, we choose along K a local orthonormal frame field $\big((E_a)_{a=1,\cdots,k}, (E_i)_{i=1,\cdots,N}\big)$ which is oriented. At points of K, we have the natural splitting

$$\mathbb{R}^n = T_pK \oplus N_pK$$

where T_pK and N_pK are respectively the tangent space to K and the normal space of K at p, which are spanned respectively by $(E_a)_a$ and $(E_i)_i$.

We denote by ∇ the connection induced by the metric $\bar{g}$ and by ∇^N the corresponding normal connection on the normal bundle. Given $p \in K$, we use some geodesic coordinates y centered at p. We also assume that at p the normal vectors $(E_i)_i$, $i = 1, \dots, N$, are transported parallely (with respect to ∇^N) through geodesics from p, so in particular

$$\bar{g}\,(\nabla_{E_a} E_j\,, E_i) = 0 \ \text{ at } p, \quad \forall\, i, j = 1, \dots, N, \ a = 1, \dots, k. \tag{2.4}$$

In a neighborhood of p in K, we consider normal geodesic coordinates

$$f(\bar{y}) := \exp_p^K(y^a E_a), \quad \forall\, \bar{y} := (y^1, \dots, y^k),$$

where $\exp^K$ is the exponential map on K and summation over repeated indices is understood. This yields the coordinate vector fields $X_a := f_*(\partial_{y^a})$. We extend the E_i along each geodesic $\gamma_E(s)$ so that they are parallel with respect to the induced connection on the normal bundle NK. This yields an orthonormal frame field X_i for NK in a neighborhood of p in K which satisfies

$$\nabla_{X_a} X_i|_p \in T_pK.$$

A coordinate system in a neighborhood of p in Ω is now defined by

$$F(\bar{y}, \bar{x}) := f(y) + x^i X_i(f(y)) \qquad \forall\, (\bar{y}, \bar{x}) := (y^1, \dots, y^k, x^1, \dots, x^N), \tag{2.5}$$

with corresponding coordinate vector fields

$$X_i := F_*(\partial_{x^i}) \quad \text{and} \quad X_a := F_*(\partial_{y^a}).$$

By our choice of coordinates, on K the metric $\bar{g}$ splits in the following way

$$\bar{g}(q) = \bar{g}_{ab}(q)\, dy^a \otimes dy^b + \bar{g}_{ij}(q)\, dx^i \otimes dx^j, \quad \forall q \in K. \tag{2.6}$$

We denote by $\Gamma_a^b(\cdot)$ the 1-forms defined on the normal bundle, NK, of K by the formula

$$\bar{g}_{bc}\Gamma_{ai}^c := \bar{g}_{bc}\Gamma_a^c(X_i) = \bar{g}(\nabla_{X_a} X_b, X_i) \quad \text{at } q = f(\bar{y}). \tag{2.7}$$

Define $q = f(\bar{y}) = F(\bar{y}, 0) \in K$ and let $(\tilde{g}_{ab}(y))$ be the induced metric on K. When we consider the metric coefficients in a neighborhood of K, we obtain a deviation from formula (2.6), which is expressed by the next lemma. The proof is somewhat standard and is thus omitted, we refer to [13, 29, 27] for more general setting.

Lemma 2.1. *At the point $F(\bar{y}, \bar{x})$, the following expansions hold, for any $a = 1, ..., k$ and any $i, j = 1, ..., N$, where $N = n - k$,*

$$\begin{aligned} \bar{g}_{ab} &= \tilde{g}_{ab} - \left\{ \tilde{g}_{ac}\, \Gamma_{bi}^c + \tilde{g}_{bc}\, \Gamma_{ai}^c \right\} \bar{x}^i + \left[\tilde{g}_{cd} \Gamma_{as}^c\, \Gamma_{bl}^d \right] \bar{x}^s \bar{x}^l \\ \bar{g}_{ij} &= \delta_{ij}; \\ \bar{g}_{aj} &= 0. \end{aligned}$$

Define $K_\varepsilon := \frac{1}{\varepsilon} K$ and $\Omega_\varepsilon := \frac{1}{\varepsilon}\Omega$. Since $F(\bar{y}, \bar{x})$ is a Fermi coordinate system on Ω, then $F_\varepsilon(y, x) := \frac{1}{\varepsilon} F(\varepsilon y, \varepsilon x)$ defines a Fermi coordinate system on Ω_ε. With this notation, here and in the sequel, by slight abuse of notation we denote $\psi(\varepsilon y, \varepsilon x)$ to actually mean $\psi(\varepsilon z) = \psi(F(\varepsilon y, \varepsilon x))$ in the Fermi coordinate system. The same way is understood to its derivatives with respect to y and x.

Given a smooth normal vector field Φ defined on K and define $x = \xi + \Phi(\varepsilon y)$ so that (y, ξ) is the Fermi coordinate system for the submanifold K_Φ. The parameter Φ will be adjusted later to show that there are solutions concentrating on K_Φ for some subsequence of ε.

We denote by $g_{\alpha\beta}$ the metric coefficients in the new coordinates (y, ξ). It follows that

$$g_{\alpha\beta} = \sum_{\gamma,\delta} \bar{g}_{\gamma\delta}\, \frac{\partial z_\alpha}{\partial \xi_\gamma}\, \frac{\partial z_\beta}{\partial \xi_\delta}.$$

Which yields

$$g_{ij} = \bar{g}_{ij}|_{\xi+\Phi}, \qquad g_{aj} = \bar{g}_{aj}|_{\xi+\Phi} + \varepsilon\, \partial_{\bar{a}} \Phi^l \bar{g}_{jl}|_{\xi+\Phi},$$

and

$$g_{ab} = \bar{g}_{ab}|_{\xi+\Phi} + \varepsilon \left\{ \bar{g}_{aj}\, \partial_{\bar{b}} \Phi^j + \bar{g}_{bj}\, \partial_{\bar{a}} \Phi^j \right\}|_{\xi+\Phi} + \varepsilon^2\, \partial_{\bar{a}} \Phi^i\, \partial_{\bar{b}} \Phi^j\, \bar{g}_{ij}|_{\xi+\Phi}$$

where summations over repeated indices is understood.

To express the error terms, it is convenient to introduce some notations. For a positive integer q, we denote by $R_q(\xi)$, $R_q(\xi,\Phi)$, $R_q(\xi,\Phi,\nabla\Phi)$, and $R_q(\xi,\Phi,\nabla\Phi,\nabla^2\Phi)$ error terms such that the following bounds hold for some positive constants C and d:

$$|R_q(\xi)| \le C\varepsilon^q(1+|\xi|^d),$$

$$|R_q(\xi,\Phi)| \le C\varepsilon^q(1+|\xi|^d),$$
$$|R_q(\xi,\Phi) - R_q(\xi,\bar\Phi)| \le C\varepsilon^q(1+|\xi|^d)|\Phi-\bar\Phi|,$$

$$|R_q(\xi,\Phi,\nabla\Phi)| \le C\varepsilon^q(1+|\xi|^d),$$
$$|R_q(\xi,\Phi,\nabla\Phi) - R_q(\xi,\bar\Phi,\nabla\bar\Phi)| \le C\varepsilon^q(1+|\xi|^d)\big(|\Phi-\bar\Phi| + |\nabla\Phi-\nabla\bar\Phi|\big),$$

and

$$|R_q(\xi,\Phi,\nabla\Phi,\nabla^2\Phi)| \le C\varepsilon^q(1+|\xi|^d) + C\varepsilon^{q+1}(1+|\xi|^d)|\nabla^2\Phi|,$$

$$\begin{aligned}&\big|R_q(\xi,\Phi,\nabla\Phi,\nabla^2\Phi) - R_q(\xi,\bar\Phi,\nabla\bar\Phi,\nabla^2\bar\Phi)\big|\\ &\le C\varepsilon^q(1+|\xi|^d)\big(|\Phi-\bar\Phi| + |\nabla\Phi-\nabla\bar\Phi|\big)\big(1+\varepsilon|\nabla^2\Phi| + \varepsilon|\nabla^2\bar\Phi|\big)\\ &\quad + C\varepsilon^{q+1}(1+|\xi|^d)|\nabla^2\Phi - \nabla^2\bar\Phi|.\end{aligned}$$

Using the expansion of the previous lemma, one can easily show that the following lemma holds true.

Lemma 2.2. *In the coordinate (y,ξ), the metric coefficients satisfy*

$$g_{ab} = \tilde g_{ab} - \varepsilon\left\{\tilde g_{bc}\Gamma^c_{ak} + \tilde g_{ac}\Gamma^c_{bk}\right\}(\xi^k+\Phi^k) + \varepsilon^2\tilde g_{cd}\,\Gamma^c_{ak}\Gamma^d_{bl}(\xi^k+\Phi^k)(\xi^l+\Phi^l) + \varepsilon^2\partial_{\bar a}\Phi^j\partial_{\bar b}\Phi^j,$$
$$g_{aj} = \varepsilon\partial_{\bar a}\Phi^j,$$
$$g_{ij} = \delta_{ij}.$$

And the inverse metric coefficients $g^{\alpha\beta}$ satisfy

$$\begin{aligned}g^{ab} &= \tilde g^{ab} + \varepsilon\left\{\tilde g^{cb}\,\Gamma^a_{ci} + \tilde g^{ca}\,\Gamma^b_{ci}\right\}(\xi^i+\Phi^i)\\ &\quad + \varepsilon^2\left(\tilde g^{ac}\,\Gamma^b_{dk}\Gamma^d_{cl} + \tilde g^{bc}\,\Gamma^a_{dk}\Gamma^d_{cl} + \tilde g^{cd}\,\Gamma^a_{dk}\Gamma^b_{cl}\right)(\xi^k+\Phi^k)(\xi^l+\Phi^l) + R_3(\xi,\Phi,\nabla\Phi),\end{aligned}$$

$$g^{aj} = -\varepsilon\,\tilde g^{ab}\,\partial_{\bar b}\Phi^j + \varepsilon^2\partial_{\bar b}\Phi^j\left\{\tilde g^{bc}\,\Gamma^a_{ci} + \tilde g^{ac}\,\Gamma^b_{ci}\right\}(\xi^i+\Phi^i) + R_3(\xi,\Phi,\nabla\Phi),$$

$$g^{ij} = \delta_{ij} + \varepsilon^2\,\tilde g^{ab}\,\partial_{\bar a}\Phi^i\partial_{\bar b}\Phi^j + R_3(\xi,\Phi,\nabla\Phi).$$

Furthermore, we have the validity of the following expansion for the log of the determinant of g:

$$\log\big(\det g\big) = \log\big(\det\tilde g\big) - 2\varepsilon\,\Gamma^b_{bk}(\xi^k+\Phi^k) - \varepsilon^2\,\Gamma^c_{am}\Gamma^a_{cl}(\xi^m+\Phi^m)(\xi^l+\Phi^l) + R_3(\xi,\Phi,\nabla\Phi).$$

Proof. The expansions of the metric in the above lemma follow from Lemma 2.1 while the expansion of the log of the determinant of g follows from the fact that one can write $g = G + M$ with

$$G = \begin{pmatrix}\tilde g & 0\\ 0 & Id_{\mathbb{R}^N}\end{pmatrix} \quad\text{and}\quad M = \mathcal{O}(\varepsilon),$$

then we have the following expansions

$$g^{-1} = \left(I + G^{-1}M\right)^{-1}G^{-1} = \left(I - G^{-1}M + G^{-1}MG^{-1}M + \mathcal{O}(\|M\|^3)\right)G^{-1}$$

and

$$\log\big(\det g\big) = \log\big(\det G\big) + \mathrm{tr}(G^{-1}M) - \frac{1}{2}\mathrm{tr}\Big((G^{-1}M)^2\Big) + \mathcal{O}(\|M\|^3).$$

and the lemma follows at once. □

2.3. **Expansion of the Laplace-Beltrami operator.** In terms the above notations, we have the following expansion of the Laplace-Beltrami operator. We postpone the proof to Appendix A.

Proposition 2.1. *Let u be a smooth function on Ω_ε. Then in the Fermi coordinate system (y,ξ), the following expansion holds*

$$\begin{aligned}
\Delta_g u &= \partial^2_{ii}u + \Delta_{K_\varepsilon}u - \varepsilon\,\Gamma^b_{bj}\partial_j u - 2\varepsilon\,\widetilde{g}^{ab}\,\partial_{\bar b}\Phi^j\,\partial^2_{aj}u + 2\,\varepsilon\,\widetilde{g}^{cb}\,\Gamma^a_{cs}\,(\xi^s+\Phi^s)\partial^2_{ab}u \\
&\quad + \varepsilon^2\,\nabla_K\Phi^i\cdot\nabla_K\Phi^j\,\partial^2_{ij}u - \varepsilon^2\,\Gamma^d_{dk}\,\partial_{\bar b}\Phi^k\,\widetilde{g}^{ab}\partial_a u + 2\varepsilon^2\partial_{\bar b}\Phi^j\left\{\widetilde{g}^{bc}\,\Gamma^a_{ci} + \widetilde{g}^{ac}\,\Gamma^b_{ci}\right\}(\xi^i+\Phi^i)\,\partial^2_{aj}u \\
&\quad + \varepsilon^2\left\{\widetilde{g}^{ac}\,\Gamma^b_{dk}\Gamma^d_{cl} + \widetilde{g}^{bc}\,\Gamma^a_{dk}\Gamma^d_{cl} + \widetilde{g}^{cd}\,\Gamma^a_{dk}\Gamma^b_{cl}\right\}(\xi^k+\Phi^k)(\xi^l+\Phi^l)\,\partial^2_{ab}u - \Gamma^c_{ak}\Gamma^a_{cj}\,(\xi^k+\Phi^k)\partial_j u \\
&\quad - \varepsilon^2\Delta_K\Phi^j\partial_j u + 2\varepsilon^3\partial^2_{\bar a\bar b}\Phi^j\Gamma^b_{ak}(\xi^k+\Phi^k)\partial_j u - \varepsilon^2\left(\widetilde{g}^{ab}\,\partial_{\bar a}\Gamma^d_{dk} - \partial_{\bar a}\{\widetilde{g}^{cb}\Gamma^a_{ck} + \widetilde{g}^{ca}\Gamma^b_{ck}\}\right)(\xi^k+\Phi^k)\partial_b u \\
&\quad + 2\varepsilon^2\left\{\widetilde{g}^{cb}\,\Gamma^a_{ci} + \widetilde{g}^{ca}\,\Gamma^b_{ci}\right\}\partial_{\bar b}\,\Phi^i\,\partial_a u + \frac{1}{2}\,\varepsilon^2\,\partial_{\bar a}(\log\det\widetilde{g})\,\{\widetilde{g}^{cb}\Gamma^a_{ci} + \widetilde{g}^{ca}\Gamma^b_{ci}\}(\xi^i+\Phi^i)\partial_b u \\
&\quad + R_3(\xi,\Phi,\nabla\Phi,\nabla^2\Phi)(\partial_j u + \partial_a u) + R_3(\xi,\Phi,\nabla\Phi)(\partial^2_{ij}u + \partial^2_{aj}u + \partial^2_{ab}u).
\end{aligned}$$

Remark 2.1. *Notice that the coefficients of all the derivatives of u in the above expansion are smooth bounded functions of the variable $\bar y = \varepsilon y$. The slow dependence of theses coefficients of y is important in our construction of some proper approximate solutions.*

3. Construction of approximate solutions

The first key step in proving Theorem 1.1 is to construct some proper approximate solutions. To achieve this goal, we first construct some very accurate local approximate solutions in a tubular neighbourhood of K_ε by an iterative scheme of Picard's type and to define some proper global approximate solutions by the gluing method.

We first recall the following key lemma and we refer to Theorem 2.1 in [7] for the proof.

Lemma 3.1. *There is an $\epsilon_0 > 0$, such that for each $\epsilon \in (0,\epsilon_0]$, (1.2) has a solution $\underline{u_\epsilon}$, such that $0 > \underline{u_\epsilon} > -\psi^{\frac{1}{p}}, \forall x \in \Omega$, and*

$$\underline{u_\epsilon}(x) = -\psi^{\frac{1}{p}}(x) - \epsilon^2\frac{\Delta\psi^{\frac{1}{p}}(x)}{p\psi^{\frac{p-1}{p}}(x)} + \ o(\epsilon^2) \tag{3.1}$$

where $\epsilon^{-2}o(\epsilon^2) \longrightarrow 0$, uniformly on any compact subset of Ω as $\epsilon \to 0$.

3.1. **Facts on the limit equation.** Recall that by the scaling, equation (1.2) (with $g(u) = |u|^p$) becomes

$$\left\{\begin{array}{rcllc} -\Delta v &=& |v|^p - \psi(\varepsilon z), & \text{in} & \Omega_\varepsilon, \\ u &=& 0 & \text{on} & \partial\Omega_\varepsilon, \end{array}\right. \tag{3.2}$$

where $\Omega_\epsilon := \Omega/\epsilon$ and $v(z) = u(\epsilon z)$.

Now given $\underline{u_\epsilon}$ be the solution of (1.2) given by the above lemma, we look for a solution of (3.2) of the form

$$\tilde v(z) = v(z) + \underline{u_\epsilon}(\epsilon z). \tag{3.3}$$

Then we can easily check that v satisfies the equation

$$-\Delta_g v = |v - \mathbf{q}(\epsilon z)|^p - |\mathbf{q}(\epsilon z)|^p \quad \text{in} \quad \Omega_\varepsilon; \qquad v = 0 \quad \text{on} \quad \partial\Omega_\varepsilon \tag{3.4}$$

where

$$\mathbf{q}(\epsilon z) = \mathbf{q}(\epsilon y, \epsilon x) := -\underline{u_\epsilon}(F(\epsilon y, \epsilon x)). \tag{3.5}$$

We consider a further change of variable in (3.4) that remplaces the main order of the potentiel $\mathbf{q}$ by $\mathbf{1}$.

Writing $\psi(\varepsilon z) = \psi(\varepsilon y, \varepsilon x)$ with $x = \xi + \Phi(\varepsilon y)$, the following expansion (of the eigenfunction ψ) hold true

$$\psi(\varepsilon z) = \psi(\varepsilon y, 0) + \varepsilon\langle \nabla^N \psi(\varepsilon y, 0), \xi + \Phi\rangle + \frac{\varepsilon^2}{2}(\nabla^N)^2\psi(\varepsilon y, 0)[\xi + \Phi]^2 + R_3(\xi, \Phi). \tag{3.6}$$

Using the expansion of the Laplace-Beltrami operator in Proposition 2.1 and the above expansion of ψ, then since the profile of solutions depends slowly on the variable y, the leading equation is given by

$$\sum_{i=1}^{N} \partial^2_{\xi_i\xi_i} v - \psi(\varepsilon y, 0) + |v|^p = 0. \tag{3.7}$$

Recalling the definition of $\mathbf{q}$ given in (3.5) we define the following functions

$$\begin{cases} \mu(\varepsilon y) := |\mathbf{q}(\epsilon y, 0)|^{\frac{p-1}{2}}, \qquad h(\varepsilon y) := \mathbf{q}(\epsilon y, 0) \qquad \forall\, y \in K_\varepsilon \\ \text{and} \\ \widetilde{\mu}(\varepsilon y) := \psi^{\frac{p-1}{2p}}(\varepsilon y, 0); \quad \widetilde{h}(\varepsilon y) := \psi^{\frac{1}{p}}(\varepsilon y, 0), \qquad \forall\, y \in K_\varepsilon. \end{cases} \tag{3.8}$$

Now, we define the following scaling

$$v(y, \xi) = h(\varepsilon y) w(y, \bar{\xi}) \quad \text{with } \bar{\xi} = \mu(\varepsilon y)\xi \in \mathbb{R}^N. \tag{3.9}$$

We will establish the expression of equation (3.4) in the new coordinates $(y, \bar{\xi})$. We turn to the equation (3.2), in the spirit of above argument, we look for a solution v of the form (3.9). An easy computation shows that

$$\begin{aligned}
&\partial_a v = h\,\partial_a w + \varepsilon(\partial_{\bar{a}} h) w + \varepsilon\, h\, \partial_{\bar{a}}\mu\, \xi^j \partial_j w, \\
&\partial^2_{ij} v = h\, \mu^2\, \partial^2_{ij} w, \\
&\partial^2_{aj} v = \varepsilon\left(\mu \partial_{\bar{a}} h + h\partial_{\bar{a}}\mu\right)\partial_j w + \, h\, \mu\, \partial^2_{aj} w + \varepsilon\, h\, \mu\, \xi^i\, \partial_{\bar{a}}\mu\, \partial^2_{ij} w, \\
&\partial^2_{ab} v = h\, \partial^2_{ab} w + \varepsilon\left(\partial_{\bar{b}} h\, \partial_a w + \partial_{\bar{a}} h\, \partial_b w + h\partial_{\bar{b}}\mu\, \xi^j \partial^2_{aj} w + h\partial_{\bar{a}}\mu\, \xi^j \partial^2_{bj} w\right) \\
&\qquad + \varepsilon^2\left(\partial_{\bar{a}} h \partial_{\bar{b}}\mu \xi^j \partial_j w + \partial_{\bar{b}} h \partial_{\bar{a}}\mu \xi^j \partial_j w + \partial^2_{\bar{a}\bar{b}} h w + h\partial_{\bar{a}}\mu\partial_{\bar{b}}\mu \xi^i \xi^j \partial^2_{ij} w + h\partial^2_{\bar{a}\bar{b}}\mu \xi^j \partial_j w\right),
\end{aligned}$$

and

$$\begin{aligned}
\Delta_{K_\varepsilon} v = {} & \varepsilon^2 \Delta_K h\, w + h\, \Delta_{K_\varepsilon} w + 2\varepsilon\, \nabla_K h \cdot \nabla_{K_\varepsilon} w + \varepsilon^2\left(h\, \Delta_K \mu + 2\, \nabla_K h \cdot \nabla_K \mu\right)\xi^j\, \partial_j w \\
& + \varepsilon^2\, h\, |\nabla_K \mu|^2\, \xi^j \xi^l\, \partial^2_{jl} w + 2\varepsilon\, h\, \xi^j\, \nabla_K \mu \cdot (\nabla_{K_\varepsilon} \partial_j w).
\end{aligned}$$

Using the above computations, one can easily see that the Laplace-Beltrami operator on v can be expanded in terms of w as

$$\underbrace{h^{-1}\mu^{-2}}_{h^{-p}}\,\Delta_g v = \Delta_{\mathbb{R}^N} w + \mu^{-2}\,\Delta_{K_\varepsilon} w + B(w),$$

with $B(w) = B_1(w) + B_2(w)$. Here B_1 and B_2 are respectively given by

$$\begin{aligned} B_1(w) &= -\varepsilon\,\mu^{-1}\,\Gamma_{bj}^b\,\partial_j w - \varepsilon^2\,\mu^{-1}\,\Gamma_{ak}^c\Gamma_{cj}^a\,(\frac{1}{\mu}\bar{\xi}^k + \Phi^k)\partial_j w + \varepsilon^2\,h^{-1}\,\mu^{-2}\,\Delta_K h\,w \\ &+ 2\varepsilon^2\,(h\,\mu^2)^{-1}\,\nabla_K h\cdot\Big(\frac{\bar{\xi}^j}{\mu}\,\nabla_K\mu - \mu\,\nabla_K\Phi^j\Big)\,\partial_j w + 2\varepsilon\;h^{-1}\,\mu^{-2}\,\nabla_K h\cdot\nabla_{K_\varepsilon} w \\ &+ \varepsilon^2\,\Big(\mu^{-2}\bar{\xi}^i\,\nabla_K\mu - \nabla_K\Phi^i\Big)\,\Big(\mu^{-2}\bar{\xi}^j\,\nabla_K\mu - \nabla_K\Phi^j\Big)\,\partial_{ij}^2 w \\ &+ \varepsilon^2\;\mu^{-2}\,\Big(\frac{\bar{\xi}^j}{\mu}\,\Delta_K\mu - 2\,\nabla_K\mu\cdot\nabla_K\Phi^j - \mu\,\Delta_K\Phi^j\Big)\,\partial_j w \\ &+ 2\varepsilon\,\mu^{-2}\,\Big(\frac{\bar{\xi}^j}{\mu}\,\nabla_K\mu - \mu\;\nabla_K\Phi^j\Big)\cdot\nabla_{K_\varepsilon}(\partial_j w), \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} h\mu^2 B_2(w) &= -\varepsilon^2\,h\,\Gamma_{dj}^d\,\nabla_K\Phi^j\;\cdot\nabla_{K_\varepsilon} w \\ &+ 2\,\varepsilon\,\widetilde{g}^{cb}\,\Gamma_{cs}^a\,\Big(\frac{1}{\mu}\,\bar{\xi}^s + \Phi^s\Big)\,\Big(h\,\partial_{ab}^2 w + \varepsilon\,\Big\{\partial_{\bar{b}} h\,\partial_a w + \partial_{\bar{a}} h\,\partial_b w + h\partial_{\bar{b}}\mu\,\frac{\bar{\xi}^j}{\mu}\partial_{aj}^2 w + h\partial_{\bar{a}}\mu\,\frac{\bar{\xi}^j}{\mu}\,\partial_{bj}^2 w\Big\}\Big) \\ &+ 2\varepsilon^2\,h\,\mu\,\partial_b\Phi^j\,\Big\{\widetilde{g}^{bc}\,\Gamma_{ci}^a + \widetilde{g}^{ac}\,\Gamma_{ci}^b\Big\}\,\Big(\frac{1}{\mu}\bar{\xi}^i + \Phi^i\Big)\,\partial_{aj}^2 w + 2\varepsilon^3\,h\,\mu\,\partial_{\bar{a}\bar{b}}^2\Phi^j\Gamma_{ak}^b\Big(\frac{1}{\mu}\bar{\xi}^k + \Phi^k\Big)\partial_j w \\ &+ \varepsilon^2 h\,\Big\{2\widetilde{g}^{ac}\,\Gamma_{dk}^b\Gamma_{cl}^d + \widetilde{g}^{cd}\,\Gamma_{dk}^a\Gamma_{cl}^b\Big\}\,\Big(\frac{1}{\mu}\bar{\xi}^k + \Phi^k\Big)\Big(\frac{1}{\mu}\bar{\xi}^l + \Phi^l\Big)\,\partial_{ab}^2 w \\ &- \varepsilon^2\,h\,\Big(\widetilde{g}^{ab}\,\partial_{\bar{a}}\Gamma_{dk}^d - \partial_{\bar{a}}\Big\{\widetilde{g}^{cb}\Gamma_{ck}^a + \widetilde{g}^{ca}\Gamma_{ck}^b\Big\}\Big)\,\Big(\frac{1}{\mu}\bar{\xi}^k + \Phi^k\Big)\partial_b w \\ &+ 2\varepsilon^2\,h\,\Big\{\widetilde{g}^{cb}\,\Gamma_{ci}^a + \widetilde{g}^{ca}\,\Gamma_{ci}^b\Big\}\,\partial_{\bar{b}}\,\Phi^i\,\partial_a w + \frac{1}{2}\,\varepsilon^2\,h\,\partial_{\bar{a}}(\log\det\widetilde{g})\,\Big\{\widetilde{g}^{cb}\Gamma_{ci}^a + \widetilde{g}^{ca}\Gamma_{ci}^b\Big\}\Big(\frac{1}{\mu}\bar{\xi}^i + \Phi^i\Big)\partial_b w \\ &+ R_3(\xi,\Phi,\nabla\Phi,\nabla^2\Phi)\Big(\partial_j w + \partial_a w\Big) + R_3(\xi,\Phi,\nabla\Phi)\Big(\partial_{ij}^2 w + \partial_{aj}^2 w + \partial_{ab}^2 w\Big). \end{aligned} \tag{3.11}$$

We set

$$S_\varepsilon(v) = \Delta_g v + |v - \mathbf{q}(\epsilon z)|^p - |\mathbf{q}(\epsilon z)|^p,$$

then by the above expansions one can easily write

$$\begin{aligned} h^{-1}\mu^{-2}\,S_\varepsilon(v) &= \Delta_{\mathbb{R}^N} w + \mu^{-2}\,\Delta_{K_\varepsilon} w + B(w) + h^{-p}(|hw - \mathbf{q}(\epsilon x,\epsilon y)|^p - |\mathbf{q}(\epsilon x,\epsilon y)|^p) \\ &= \Delta_{\mathbb{R}^N} w + \mu^{-2}\,\Delta_{K_\varepsilon} w + B(w) + h^{-p}(|h(1-w) + (\mathbf{q}(\epsilon x,\epsilon y) - h)|^p - |\mathbf{q}(\epsilon x,\epsilon y)|^p). \end{aligned}$$

Now using the following expansion of the potential $\mathbf{q}$:

$$\mathbf{q}(\varepsilon y,\varepsilon x) = \mathbf{q}(\varepsilon y,0) + \varepsilon\langle\nabla^N\mathbf{q}(\varepsilon y,0),\frac{\bar{\xi}}{\mu} + \Phi\rangle + \frac{\varepsilon^2}{2}(\nabla^N)^2\mathbf{q}(\varepsilon y,0)[\frac{\bar{\xi}}{\mu} + \Phi]^2 + R_3(\bar{\xi},\Phi),$$

we obtain

$$
\begin{aligned}
|h(1-w)+(\mathbf{q}(\epsilon x,\epsilon y)-h)|^p &= h^p|1-w|^p + ph^{p-1}|1-w|^{p-2}(1-w)(\mathbf{q}(\epsilon x,\epsilon y)-h)\\
&\quad + \frac{p(p-1)}{2}h^{p-2}|1-w|^{p-2}(\mathbf{q}-h)^2 + \mathcal{O}(\varepsilon^3)\\
&= h^p\Big[|1-w|^p + ph^{-1}|1-w|^{p-2}(1-w)\Big(\varepsilon < \nabla^N\mathbf{q}(\epsilon y,0),\xi+\Phi >\\
&\quad + \frac{1}{2}\varepsilon^2(\nabla^N)^2\mathbf{q}(\epsilon y,0)[\xi+\Phi]^2\Big)\\
&\quad + \frac{p(p-1)}{2}h^{-2}|1-w|^{p-2}\epsilon^2 < \nabla^N\mathbf{q}(\epsilon y,0),\xi+\Phi >^2 + \mathcal{O}(\varepsilon^3)\Big]\\
&= h^p\Big[|1-w|^p - ph^{-1}|1-w|^{p-2}(w-1)\varepsilon < \nabla^N\mathbf{q}(\epsilon y,0),\xi+\Phi >\\
&\quad - \frac{1}{2}ph^{-1}|1-w|^{p-2}(w-1)\varepsilon^2(\nabla^N)^2\mathbf{q}(\epsilon y,0)[\xi+\Phi]^2\\
&\quad + \frac{p(p-1)}{2}h^{-2}|1-w|^{p-2}\epsilon^2 < \nabla^N\mathbf{q}(\epsilon y,0),\xi+\Phi >^2 + \mathcal{O}(\varepsilon^3)\Big].
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
|\mathbf{q}|^p &= |h+\varepsilon\langle\nabla^N\mathbf{q}(\varepsilon y,0),\frac{\bar{\xi}}{\mu}+\Phi\rangle + \frac{\varepsilon^2}{2}(\nabla^N)^2\mathbf{q}(\varepsilon y,0)[\frac{\bar{\xi}}{\mu}+\Phi]^2 + R_3(\bar{\xi},\Phi)|^p\\
&= h^p\Big[1+ph^{-1}\epsilon < \nabla^N\mathbf{q}(\epsilon y,0),\xi+\Phi > + \frac{1}{2}ph^{-1}\varepsilon^2(\nabla^N)^2\mathbf{q}(\epsilon y,0)[\xi+\Phi]^2\\
&\quad + \frac{p(p-1)}{2}\varepsilon^2h^{-2} < \nabla^N\mathbf{q}(\epsilon y,0),\xi+\Phi >^2 + \mathcal{O}(\varepsilon^3)\Big].
\end{aligned}
$$

So we have

$$
\begin{aligned}
|h(1-w)+&(\mathbf{q}(\epsilon x,\epsilon y)-h)|^p - |\mathbf{q}(\epsilon x,\epsilon y)|^p = h^p\Big[|w-1|^p - 1 + ph^{-1}\epsilon < \nabla^N\mathbf{q}(\epsilon y,0),\xi+\Phi > \times\\
&\times\Big(|1-w|^{p-2}(1-w)-1\Big) + \frac{1}{2}ph^{-1}\varepsilon^2(\nabla^N)^2\mathbf{q}(\epsilon y,0)[\xi+\Phi]^2\Big(|1-w|^{p-2}(1-w)-1\Big)\\
&+ \frac{p(p-1)}{2}\varepsilon^2h^{-2} < \nabla^N\mathbf{q}(\epsilon y,0),\xi+\Phi >^2\Big(|1-w|^{p-2}-1\Big) + \mathcal{O}(\varepsilon^3)\Big].
\end{aligned}
$$

We have the equation satisfied by w in the new coordinates

$$
\widetilde{S}_\varepsilon(w) := \Delta_{\mathbb{R}^N}w + h^{1-p}\,\Delta_{K_\varepsilon}w + |w-1|^p - 1 + \widetilde{B}(w) = 0 \tag{3.12}
$$

where $\widetilde{B}(w) = \widetilde{B}_1(w) + \widetilde{B}_2(w)$, with

$$
\begin{aligned}
\widetilde{B}_1(w) &= B_1(w) + ph^{-1}\epsilon < \nabla^N\ \mathbf{q}(\epsilon y,0),\frac{\bar{\xi}}{\mu}+\ \Phi > (|1-w|^{p-2}(1-w)-1)\\
&\quad + \frac{1}{2}ph^{-1}\epsilon^2(\nabla^N)^2\ \mathbf{q}(\varepsilon y,0)[\frac{\bar{\xi}}{\mu}+\ \Phi]^2\Big(|1-w|^{p-2}(1-w)-1\Big)\\
&\quad + \frac{p(p-1)}{2}h^{-2}\epsilon^2 < \nabla^N\ \mathbf{q}(\varepsilon y,0),\frac{\bar{\xi}}{\mu}+\Phi >^2\Big(|1-w|^{p-2}-1\Big),
\end{aligned}
$$

and

$$
\widetilde{B}_2(w) = B_2(w) + R_3(\bar{\xi},\Phi).
$$

Where $B_1(w)$ and $B_2(w)$ are defined respectively in (3.10) and (3.11).
At the end of this subsection, let us list some basic and useful properties of positive solutions of the limit equation (3.7).

Consider the following elliptic problem:

$$\begin{cases} -\Delta_{\mathbb{R}^N} u = |u-1|^p - 1, & u > 0 \text{ in } \mathbb{R}^N \\ u(0) = \max_{\xi\in\mathbb{R}^N} u(\xi) \\ u \in H^1(\mathbb{R}^N) \end{cases} \tag{3.13}$$

Since p is subcritical in $\mathbb{R}^N$, using the standard Lions's concentration compactness arguments, we can prove that (3.13) have a positive solution U. It is easy to see that U decays exponentially at infinity and is radially symmetric.
The following Proposition is essential for us to construct solutions.

Proposition 3.1. *Let U be a solution of (3.13). If $1 < p < \infty$ for $N = 2$ and $1 < p < \frac{N+2}{N-2}$ for $N \geq 3$, then U is unique and nondegenerate. That is, the kernel of the operator $-\Delta u - p|U-1|^{p-2}(U-1)u$ in $H^1(\mathbb{R}^N)$ is spanned by $\{\frac{\partial U}{\partial x_1}, ..., \frac{\partial U}{\partial x_N}\}$.*

Then every solution of problem (3.13) has the form $U(\cdot - Q)$ for some $Q \in \mathbb{R}^N$, where $U(x) = U(|x|) \in C^\infty(\mathbb{R}^N)$ is the unique positive radial solution which satisfies

$$\lim_{r\to\infty} r^{\frac{N-1}{2}} e^r U(r) = c_{N,p}, \quad \lim_{r\to\infty} \frac{U'(r)}{U(r)} = -1. \tag{3.14}$$

Here $c_{N,p}$ is a positive constant depending only on N and p. Furthermore, U is non-degenerate in the sense that

$$Ker\Big(-\Delta_{\mathbb{R}^N} - p|U-1|^{p-2}(U-1)\Big) \cap L^\infty(\mathbb{R}^N) = Span\Big\{\partial_{x_1}U, \cdots, \partial_{x_N}U\Big\},$$

and the Morse index of U is one, that is, the linear operator

$$L_0 := -\Delta_{\mathbb{R}^N} - p|U-1|^{p-2}(U-1)$$

has only one negative eigenvalue $\lambda_0 < 0$, and the unique even and positive eigenfunction corresponding to λ_0 can be denoted by Z.

Proof. It is easy to check that the uniqueness follows from the general theorem in [21]. The nondegeneracy in the space of radial functions follows easily as [5], and then the nondegeneracy in general follows easily as in [[6], p. 970, 971].

An immediate consequence of the above proposition is the following result

Corollary 3.1. *There is a constant $\gamma_0 > 0$ such that*

$$\int_{\mathbb{R}^N} \Big\{|\nabla\varphi|^2 + p|U-1|^{p-1}\varphi^2\Big\}\, d\bar{\xi} \geq \gamma_0 \int_{\mathbb{R}^N} (|\nabla\varphi|^2 + \Phi^2)\, d\bar{\xi}, \tag{3.15}$$

whenever $\varphi \in H^1(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} \varphi\, \partial_j U\, d\bar{\xi} = 0 = \int_{\mathbb{R}^N} \varphi Z\, d\bar{\xi}, \quad \forall\, j = 1, \ldots, N.$$

We are now in position to construct very accurate approximate solutions. This is the aim of the next subsections.

3.2. **Local approximate solutions.** By (3.12) it obvious that in a tubular neighbourhood of the scaled submanifold K_ε, the equation $S_\varepsilon(v) = 0$ is equivalent to $\widetilde{S}_\varepsilon(w) = 0$.

We will look for approximate solutions of the form

$$w = w(y, \bar{\xi}) = U(\bar{\xi}) + \sum_{\ell=1}^{I} \varepsilon^\ell w_\ell(\varepsilon y, \bar{\xi}) + \varepsilon e(\varepsilon y) Z(\bar{\xi}), \tag{3.16}$$

where $I \in \mathbb{N}_+$, U and Z are given in Proposition 3.1, w_ℓ's and e are smooth bounded functions on their variables.

To solve $\widetilde{S}_\varepsilon(w) = 0$ accurately, the normal section Φ is to be chosen in the following form

$$\Phi = \Phi_0 + \sum_{\ell=1}^{I-1} \varepsilon^\ell\, \Phi_\ell,$$

where $\Phi_0, \ldots, \Phi_{I-1}$ are smooth bounded functions on $\bar{y}$.

3.2.1. *Expansion at first order in ε.* We first solve the equation $\widetilde{S}_\varepsilon(w) = 0$ up to order ε. Let w be of the form (3.16), then using the fact that $\mu(\epsilon y) = \widetilde{\mu}(\epsilon y) + \mathcal{O}(\varepsilon^2)$ and

$$\nabla^N \mathbf{q}(\varepsilon y, 0) = \nabla^N(\psi^{1/p}(\varepsilon y, 0)) + \mathcal{O}(\varepsilon^2) = \frac{1}{p}\psi^{\frac{1-p}{p}} \nabla^N \psi(\varepsilon y) + \mathcal{O}(\varepsilon^2)$$

we can write

$$\begin{aligned}
\widetilde{S}_\varepsilon(w) \;&=\; \varepsilon\Big(\Delta_{\mathbb{R}^N} w_1 + p|U-1|^{p-2}(U-1)w_1\Big) + \varepsilon(\varepsilon^2 \widetilde{\mu}^{-2} \Delta_K e - \lambda_0 e) Z \\
&+\; \varepsilon\Big(-\widetilde{\mu}^{-1}\Gamma^b_{bj}\partial_j U + \widetilde{\mu}^{\frac{2}{1-p}}\,\psi^{\frac{1-p}{p}} \langle \nabla^N \psi(\varepsilon y, 0), \frac{\bar{\xi}}{\widetilde{\mu}} + \Phi_0\rangle (|1-U|^{p-2}(1-U) - 1)\Big) \\
&+\; \mathcal{O}(\varepsilon^2).
\end{aligned}$$

Our aim is to choose the function w_1 in such away the term of order ε in the right-hand side of above equation vanishes. This is clearly equivalent to choose w_1 to be solution of the following linear equation

$$L_0 w_1 = \widetilde{\mu}^{-1}\, \Gamma^b_{bj}\partial_j U - \widetilde{\mu}^{\frac{2}{1-p}}\, \psi^{\frac{1-p}{p}} \langle \nabla^N \psi(\varepsilon y, 0), \frac{\bar{\xi}}{\widetilde{\mu}} + \Phi_0\rangle (|1-U|^{p-2}(1-U) - 1) \tag{3.17}$$

where L_0 is the operator defined in Proposition 3.1. Here and in the following, we will keep the term $\varepsilon(\varepsilon^2\widetilde{\mu}^{-2}\Delta_K e - \lambda_0 e)Z$ in the error. The reason is simply that it cannot be cancelled without solving an equation for e since $L_0 Z = \lambda_0 Z$.

By Proposition 3.1, equation (3.17) is solvable if and only if the right hand side is orthogonal to the kernel of L_0. Namely, if and only if for all $i = 1, \ldots, N$,

$$\int_{\mathbb{R}^N} \Big(\widetilde{\mu}^{-1}\Gamma^b_{bj}\partial_j U - \widetilde{\mu}^{\frac{2}{1-p}} \psi^{\frac{1-p}{p}} \langle \nabla^N \psi(\varepsilon y, 0), \frac{\bar{\xi}}{\widetilde{\mu}} + \Phi_0\rangle (|1-U|^{p-2}(1-U) - 1)\Big)\partial_i U \, d\bar{\xi} = 0. \tag{3.18}$$

Observe that since U is even in $\bar{\xi}$ and $\partial_i U$ is odd in $\bar{\xi}$, we have

$$\int_{\mathbb{R}^N} \langle \nabla^N \psi^{1/p}(\varepsilon y, 0), \Phi_0\rangle\, (|1-U|^{p-2}(1-U) - 1)\partial_i U\, d\bar{\xi} = 0.$$

We conclude that equation (3.17) is solvable if and only if for all $i = 1, \ldots, N$,

$$\int_{\mathbb{R}^N} \widetilde{\mu}^{-1}\Gamma^b_{bj}\partial_j U - \widetilde{\mu}^{\frac{2}{1-p}}\, \psi^{\frac{1-p}{p}} \langle \nabla^N \psi(\varepsilon y, 0), \frac{\bar{\xi}}{\widetilde{\mu}}\rangle\, (|1-U|^{p-2}(1-U) - 1)\Big)\partial_i U\, d\bar{\xi} = 0. \tag{3.19}$$

Recalling the definition of $\widetilde{\mu}$ given in (3.8), we have that $\widetilde{\mu}^{\frac{2}{1-p}}\,\psi^{\frac{1-p}{p}} = \psi^{-1}$. It follows then that (3.17) is solvable if and only if for all $i = 1, \dots, N$

$$\int_{\mathbb{R}^N} \Gamma^b_{bj}\partial_j U - \psi^{-1}\,\langle\nabla^N\psi(\varepsilon y,0),\bar{\xi}\rangle\,\big(|1-U|^{p-2}(1-U)-1\big)\partial_i U\,d\bar{\xi} = 0. \tag{3.20}$$

Since U is radially symmetric, (3.20) is equivalent to

$$\Gamma^b_{bi}\int_{\mathbb{R}^N}|\partial_1 U|^2 d\bar{\xi} = \psi^{-1}\partial_i\psi(\varepsilon y,0)\int_{\mathbb{R}^N}(|1-U|^{p-2}(1-U)-1)\xi^i\partial_i U d\bar{\xi}.$$

We claim that

$$\begin{cases} \int_{\mathbb{R}^N}(|1-U|^{p-2}(1-U)-1)\xi^i\partial_i U d\bar{\xi} = N\int_{\mathbb{R}^N} U d\bar{\xi}, \\ \text{and} \\ \int_{\mathbb{R}^N} U d\bar{\xi} = \dfrac{\sigma}{N}\int_{\mathbb{R}^N}|\partial_1 U|^2 d\bar{\xi} \end{cases} \tag{3.21}$$

where $\sigma = \dfrac{p-1}{p}(\frac{p+1}{p-1} - \frac{n-k}{2})$. See Appendix B for the proof of (3.21)

We get

$$\frac{1}{\sigma}\Gamma^b_{bi}\int_{\mathbb{R}^N} U d\bar{\xi} = \psi^{-1}\partial_i\psi(\epsilon y,0)\int_{\mathbb{R}^N} U d\bar{\xi}.$$

Using the definition of the mean curvature vector H on K

$$H = (-\Gamma^b_{bi})$$

we get that equation (3.17) is solvable if and only if

$$\sigma\nabla^N\psi(\varepsilon y,0) = -\psi(\varepsilon y,0)H(\varepsilon y), \tag{3.22}$$

This is exactly our stationary condition on K. Using (3.22), the equation of w_1 becomes simply

$$L_0(w_1) = \widetilde{\mu}^{-1}\Gamma^b_{bj}\partial_j U + \sigma^{-1} <\, H,\, \frac{\bar{\xi}}{\widetilde{\mu}} + \Phi_0 > (|1-U|^{p-2}(1-U)-1).$$

This can be rewritten for convenience as

$$\begin{aligned} L_0(w_1) \;&=\; \widetilde{\mu}^{-1}\Gamma^b_{bj}\Big[\partial_j U - \sigma^{-1}\bar{\xi}^j\Big(|1-U|^{p-2}(1-U)-1\Big)\Big] \\ &+\; \sigma^{-1} < H,\Phi_0 > \Big(|1-U|^{p-2}(1-U)-1\Big) \end{aligned} \tag{3.23}$$

Hence we can write

$$w_1 = w_{1,1} + w_{1,2}, \tag{3.24}$$

with

$$w_{1,1} = \widetilde{\mu}^{-1}\,\Gamma^b_{bj}U_j \quad \text{and} \quad w_{1,2} = \sigma^{-1}\langle H,\Phi_0\rangle U_0. \tag{3.25}$$

Here U_j is the unique smooth bounded function satisfying

$$L_0 U_j = \partial_j U - \sigma^{-1}\,\bar{\xi}^j\,(|1-U|^{p-2}(1-U)-1), \quad \int_{\mathbb{R}^N} U_j\,\partial_i U\,d\bar{\xi} = 0,\ \forall\, i = 1,\dots,N, \tag{3.26}$$

while U_0 is the unique smooth bounded function such that

$$L_0 U_0 = (|1-U|^{p-2}(1-U)-1), \qquad \int_{\mathbb{R}^N} U_0\,\partial_i U\,d\bar{\xi} = 0,\ \forall\, i = 1,\dots,N. \tag{3.27}$$

It follows immediately that $w_1 = w_1(\varepsilon y, \bar\xi)$ is smooth bounded on its variable. Furthermore, it is easily seen that U_j is odd on variable $\bar\xi^j$ and is even on other variables. Moreover, U_0 has an explicit expression

$$U_0 = -\frac{1}{pN}U + \frac{1-p}{2pN}\bar\xi \cdot \nabla U. \tag{3.28}$$

3.2.2. *Expansion at second order in ε.* We will now solve the equation $\widetilde S_\varepsilon(w) = 0$ up to order ε^2 by choosing the function w_2 and the normal section Φ_0 together. Suppose w has the form (3.16), then

$$\begin{aligned}\widetilde S_\varepsilon(w) = {}& \varepsilon^2\Big(\Delta_{\mathbb{R}^N} w_2 + p|U-1|^{p-2}(U-1)w_2\Big) + \varepsilon(\varepsilon^2\tilde\mu^{-2}\Delta_K e - \lambda_0 e)Z\\ & + \varepsilon^2\mathfrak{F}_2 + \varepsilon^2\mathfrak{G}_2 + \mathcal{O}(\varepsilon^3),\end{aligned}$$

where the terms $\mathfrak{F}_2$ and $\mathfrak{G}_2$ are respectively given by

$$\begin{aligned}\mathfrak{F}_2 = {}& -\tilde\mu^{-1}\,\Gamma^b_{bj}\partial_j w_1 + \psi^{-1}\,\langle\nabla^N\psi(\varepsilon y,0),\Phi_1\rangle\,(|1-U|^{p-2}(1-U)-1)\\ & -\tilde\mu^{-1}\,\Gamma^c_{ak}\Gamma^a_{cj}\,(\frac{\bar\xi^k}{\tilde\mu} + \Phi^k_0)\partial_j U + \tilde\mu^{-2}\Big(\frac{\bar\xi^j}{\tilde\mu}\,\Delta_K\tilde\mu - 2\,\nabla_K\tilde\mu\cdot\nabla_K\Phi^j_0 - \tilde\mu\,\Delta_K\Phi^j_0\Big)\,\partial_j U\\ & + \tilde h^{-1}\tilde\mu^{-2}\,\Delta_K\tilde h\,U + 2(\tilde h\tilde\mu^2)^{-1}\nabla_K\tilde h\cdot\Big(\frac{\bar\xi^j}{\tilde\mu}\,\nabla_K\tilde\mu - \tilde\mu\,\nabla_K\Phi^j_0\Big)\,\partial_j U\\ & + \Big(\tilde\mu^{-2}\bar\xi^i\,\nabla_K\tilde\mu - \nabla_K\Phi^i_0\Big)\Big(\tilde\mu^{-2}\bar\xi^j\,\nabla_K\tilde\mu - \nabla_K\Phi^j_0\Big)\,\partial^2_{ij}U\\ & + \psi^{-1}\,\langle\nabla^N\psi, \frac{\bar\xi}{\tilde\mu} + \Phi_0\rangle((1-p)|U-1|^{p-2}w_1)\\ & + \frac{1}{2}p\,\tilde h^{-1}\,(\nabla^N)^2(\psi^{\frac{1}{p}})(\varepsilon y,0)[\frac{\bar\xi}{\tilde\mu} + \Phi_0, \frac{\bar\xi}{\tilde\mu} + \Phi_0]\,(|1-U|^{p-2}(1-U)-1)\\ & + \frac{1}{2}p(p-1)|U-1|^{p-2}w_1^2 + \frac{1}{2}p(p-1)\tilde h^{-2}\langle\nabla^N(\psi^{\frac{1}{p}}), \frac{\bar\xi}{\tilde\mu} + \Phi_0\rangle^2(|1-U|^{p-2}-1)\end{aligned}$$

and

$$\begin{aligned}\mathfrak{G}_2 = {}& -\tilde\mu^{-1}\Gamma^b_{bj}\,e\,\partial_j Z + \psi^{-1}\,((1-p)|U-1|^{p-2})\langle\nabla^N\psi, \frac{\bar\xi}{\tilde\mu} + \Phi_0\rangle eZ\\ & + \frac{1}{2}p(p-1)|U-1|^{p-2}\Big\{(w_1+eZ)^2 - w_1^2\Big\}.\end{aligned} \tag{3.29}$$

Hence the term of order ε^2 vanishes (except the term $\varepsilon(-\varepsilon^2\tilde\mu^{-2}\Delta_K e + \lambda_0 e)Z)$ if and only if w_2 satisfies the equation

$$L_0 w_2 = -\mathfrak{F}_2 - \mathfrak{G}_2.$$

By Freedholm alternative this equation is solvable if and only if $\mathfrak{F}_2 + \mathfrak{G}_2$ is L^2 orthogonal to the kernel of linearized operator L_0, which is spanned by the functions $\partial_i U$, $i = 1,\dots,N$.

It is convenient to write $\mathfrak{F}_2$ as

$$\mathfrak{F}_2 = \psi^{-1}\langle\nabla^N\psi(\varepsilon y,0),\Phi_1\rangle(|1-U|^{p-2}(1-U)-1) + \widetilde{\mathfrak{F}}_2.$$

The function $\widetilde{\mathfrak{F}}_2$ does not involve ψ and by (3.22) and arguing as for w_1, we can write w_2 as

$$w_2 = w_{2,1} + w_{2,2},$$

where $w_{2,2} = \sigma^{-1}\langle H,\psi\rangle U_0$ solves the equation

$$L_0 w_{2,2} = \sigma^{-1}\langle H,\Phi_1\rangle(|1-U|^{p-2}(1-U)-1),$$

and $w_{2,1}$ will solve the equation

$$L_0 w_{2,1} = -\widetilde{\mathfrak{F}}_2 - \mathfrak{G}_2.$$

To solve the equation on $w_{2,1}$ it is convenient to write

$$\widetilde{\mathfrak{F}}_2 = \widetilde{\mathfrak{F}}_2(\Phi_0) = S_{2,0} + S_2(\Phi_0) + N_2(\Phi_0),$$

where $S_{2,0} = \widetilde{\mathfrak{F}}_2(0)$ does not involve Φ_0, $S_2(\Phi_0)$ is the sum of linear terms of Φ_0, and $N_2(\Phi_0)$ is the nonlinear term of Φ_0.

Recall that $w_1 = w_{1,1} + w_{1,2}$ with

$$w_{1,1} = \widetilde{\mu}^{-1}\,\Gamma^b_{bj} U_j \quad \text{and} \quad w_{1,2} = \sigma^{-1}\langle H, \Phi_0\rangle U_0.$$

Then

$$\begin{aligned}
S_{2,0} &= -\widetilde{\mu}^{-1}\,\Gamma^b_{bj}\,\partial_j w_{1,1} - \widetilde{\mu}^{-2}\,\Gamma^c_{ak}\Gamma^a_{cj}(\bar{\xi}^k\,\partial_j U)\\
&+ \widetilde{\mu}^{-3}\Delta_K\widetilde{\mu}\,\bar{\xi}^j\,\partial_j U + \widetilde{h}^{-1}\widetilde{\mu}^{-2}\Delta_K\widetilde{h}\,U + 2(\widetilde{h}\widetilde{\mu}^3)^{-1}(\nabla_K\widetilde{h}\cdot\nabla_K\widetilde{\mu})\bar{\xi}^j\,\partial_j U\\
&+ \widetilde{\mu}^{-4}|\nabla_K\widetilde{\mu}|^2\,(\bar{\xi}^i\,\bar{\xi}^j\,\partial^2_{ij}U) + \psi^{-1}\,\widetilde{\mu}^{-1}\langle\nabla^N\psi(\varepsilon y,0),\bar{\xi}\rangle\big((1-p)|U-1|^{p-2}\big)w_{1,1}\\
&+ \frac{1}{2}p\widetilde{h}^{-1}\widetilde{\mu}^{-2}(\nabla^N)^2(\psi^{\frac{1}{p}})[\bar{\xi},\bar{\xi}]\,\Big(|1-U|^{p-2}(1-U)-1\Big)\\
&+ \frac{1}{2}p(p-1)|U-1|^{p-2}w_{1,1}^2 + \frac{1}{2}p(p-1)\widetilde{h}^{-2}\widetilde{\mu}^{-2}\langle\nabla^N(\psi^{\frac{1}{p}})(\varepsilon y,0),\bar{\xi}\rangle^2\Big(|1-U|^{p-2}-1\Big),
\end{aligned}$$

$$\begin{aligned}
S_2(\Phi_0) &= -\widetilde{\mu}^{-1}\Gamma^b_{bj}\,\partial_j w_{1,2} - \widetilde{\mu}^{-1}\,\Gamma^c_{ak}\Gamma^a_{cj}\Phi^k_0\,\partial_j U\\
&- \widetilde{\mu}^{-2}\Big(2\,\nabla_K\widetilde{\mu}\cdot\nabla_K\Phi^j_0 + \widetilde{\mu}\,\Delta_K\Phi^j_0\Big)\partial_j U - 2(\widetilde{h}\widetilde{\mu})^{-1}\Big(\nabla_K\widetilde{h}\cdot\nabla_K\Phi^j_0\Big)\partial_j U\\
&- 2\widetilde{\mu}^{-2}\Big(\nabla_K\widetilde{\mu}\cdot\nabla_K\Phi^j_0\Big)\,(\bar{\xi}^i\,\partial^2_{ij}U) + \widetilde{\mu}^{-1}\psi^{-1}\langle\nabla^N\psi,\bar{\xi}\rangle\big((1-p)|U-1|^{p-2}\big)w_{1,2}\\
&+ \psi^{-1}\langle\nabla^N\psi,\Phi_0\rangle\big((1-p)|U-1|^{p-2}\big)w_{1,1} + \frac{p-1}{p}\sigma^{-2}\widetilde{\mu}^{-1}\langle H,\bar{\xi}\rangle\langle H,\Phi_0\rangle\big(|U-1|^{p-2}-1\big)\\
&+ p\widetilde{h}^{-1}\widetilde{\mu}^{-1}(\nabla^N)^2(\psi^{\frac{1}{p}})[\Phi_0,\bar{\xi}]\,\Big(|U-1|^{p-2}(1-U)-1\Big) + p(p-1)|U-1|^{p-2}w_{1,1}w_{1,2}
\end{aligned}$$

and

$$\begin{aligned}
N_2(\Phi_0) &= (\nabla_K\Phi^i_0\cdot\nabla_K\Phi^j_0)\,\partial^2_{ij}U + \psi^{-1}\langle\nabla^N\psi,\Phi_0\rangle\big((1-p)|U-1|^{p-2}\big)w_{1,2}\\
&+ \frac{1}{2}\,p\,\widetilde{h}^{-1}\,(\nabla^N)^2(\psi^{\frac{1}{p}})[\Phi_0,\Phi_0]\,\big(|U-1|^{p-2}(1-U)-1\big) + \frac{1}{2}p(p-1)|U-1|^{p-2}w_{1,2}^2\\
&+ \frac{1}{2}\frac{p-1}{p}\sigma^{-2}\langle H,\Phi_0\rangle^2\big(|U-1|^{p-2}-1\big).
\end{aligned}$$

Using the stationary condition and (3.8), we have that

$$\begin{aligned}\int_{\mathbb{R}^N} S_2(\Phi_0)\,\partial_s U =& -\tilde{\mu}^{-1}\Gamma^b_{bj}\int_{\mathbb{R}^N}\partial_j w_{1,2}\,\partial_s U-\tilde{\mu}^{-1}\Gamma^c_{ak}\Gamma^a_{cj}\,\Phi^k_0\int_{\mathbb{R}^N}\partial_j U\,\partial_s U\\ &-(\tilde{\mu})^{-2}\Big(2\nabla_K\tilde{\mu}\cdot\nabla_K\Phi^j_0+\tilde{\mu}\,\Delta_K\Phi^j_0\Big)\int_{\mathbb{R}^N}\partial_j U\,\partial_s U\\ &-2(\tilde{h}\tilde{\mu})^{-1}\Big(\nabla_K\tilde{h}\cdot\nabla_K\Phi^j_0\Big)\int_{\mathbb{R}^N}\partial_j U\,\partial_s U-2\tilde{\mu}^{-2}\Big(\nabla_K\tilde{\mu}\cdot\nabla_K\Phi^j_0\Big)\int_{\mathbb{R}^N}\bar{\xi}^i\,\partial^2_{ij}U\,\partial_s U\\ &+(p-1)\sigma^{-1}H^j\Big(\tilde{\mu}^{-1}\int_{\mathbb{R}^N}|1-U|^{p-2}\bar{\xi}^j\,w_{1,2}\,\partial_s U+\Phi^j_0\int_{\mathbb{R}^N}|U-1|^{p-2}w_{1,1}\,\partial_s U\Big)\\ &+\frac{1-p}{p}\tilde{\mu}^{-1}\sigma^{-2}H^i\,H^j\Phi^j_0\int_{\mathbb{R}^N}(|1-U|^{p-2}(1-U)-1)\bar{\xi}^i\,\partial_s U\\ &+\tilde{\mu}^{-1}\psi^{-1}\partial^2_{ij}\psi\,\Phi^j_0\int_{\mathbb{R}^N}(|1-U|^{p-2}(1-U)-1)\bar{\xi}^i\,\partial_s U\\ &+p(p-1)\int_{\mathbb{R}^N}|U-1|^{p-2}\,w_{1,1}\,w_{1,2}\,\partial_s U\\ &+\frac{(p-1)}{p}\sigma^{-2}\tilde{\mu}^{-1}H^i\,H^j\Phi^j_0\int_{\mathbb{R}^N}(|1-U|^{p-2}-1)\bar{\xi}^i\,\partial_s U\end{aligned}$$

We start first by computing the integrals involving $w_{1,1}$ and $w_{1,2}$ in the above formula. Let us denote by A the sum of such terms. It can written as

$$A=\int_{\mathbb{R}^N}N_0(w_1)\partial_s U\,d\xi+\int_{\mathbb{R}^N}N_1(U,w_1)\partial_s U\,d\xi$$

where

$$N_0(w_1):=p(p-1)|U-1|^{p-2}\,w_{1,1}\,w_{1,2}$$

and

$$N_1(U,w_1):=-\tilde{\mu}^{-1}\Gamma^b_{bj}\partial_j w_{1,2}+(p-1)\sigma^{-1}H^j\left(\tilde{\mu}^{-1}\,|1-U|^{p-2}\bar{\xi}^j\,w_{1,2}+\Phi^j_0\,|U-1|^{p-2}w_{1,1}\right).$$

To compute the term A, we differentiate the equation (3.27) with respect to the variable $\bar{\xi}^j$ to obtain

$$L_0(\partial_s U_0)+p(p-1)|U-1|^{p-2}U_0\partial_s U=-(p-1)\,|1-U|^{p-2}\partial_s U. \tag{3.30}$$

Multiplying the equation (3.26) by $\partial_s U_0$ and integrating by parts, we have

$$\int_{\mathbb{R}^N}L_0(\partial_s U_0)\,U_j=\int_{\mathbb{R}^N}\Big[\underbrace{\partial_j U-\sigma^{-1}\xi^j\Big(|1-U|^{p-2}(1-U)-1\Big)}_{\mathcal{T}(U)}\Big]\partial_s U_0$$

On the other hand

$$\begin{aligned}\int_{\mathbb{R}^N}L_0(\partial_s U_0)\,U_j &= \int_{\mathbb{R}^N}\partial_s(\Delta_{\mathbb{R}^N}U_0)U_j+p\int_{\mathbb{R}^N}\{|1-U|^{p-2}(U-1)U_j\}\partial_s U_0\\ &= -\int_{\mathbb{R}^N}\Delta_{\mathbb{R}^N}U_0\partial_s U_j-p\int_{\mathbb{R}^N}U_0\Big\{(p-1)|U-1|^{p-2}\partial_s UU_j+|U-1|^{p-2}(U-1)\partial_s U_j\Big\}\\ &= -\int_{\mathbb{R}^N}L_0(U_0)\partial_s U_j-p(p-1)\int_{\mathbb{R}^N}|U-1|^{p-2}\partial_s UU_0U_j.\end{aligned}$$

It then follows that

$$\begin{aligned}\int_{\mathbb{R}^N}\mathcal{T}(U)\partial_s U_0&=\int_{\mathbb{R}^N}(|1-U|^{p-2}(1-U)-1)\partial_s U_j\;-\;p(p-1)\int_{\mathbb{R}^N}|U-1|^{p-2}\partial_s UU_0U_j\\ &=-(p-1)\int_{\mathbb{R}^N}|1-U|^{p-2}U_j\partial_s U\;-\;p(p-1)\int_{\mathbb{R}^N}|U-1|^{p-2}\partial_s UU_0U_j.\end{aligned}$$

We deduce that

$$
\begin{aligned}
\int_{\mathbb{R}^N} N_0(w_1)\partial_s U d\xi &= \widetilde{\mu}^{-1}\sigma^{-1} < H, \Phi_0 > \Gamma^a_{aj}\Big(- \int_{\mathbb{R}^N} \mathcal{T}(U)\partial_s U_0 d\xi - (p-1)\int_{\mathbb{R}^N} |1-U|^{p-2}U_j\partial_s U\Big) \\
&= -\widetilde{\mu}^{-1}\sigma^{-1} < H, \Phi_0 > \Gamma^a_{aj}\int_{\mathbb{R}^N}\Big[\partial_j U - \sigma^{-1}\xi^j(|1-U|^{p-2}(1-U)-1)\Big]\partial_s U_0 \\
&- \widetilde{\mu}^{-1}\sigma^{-1} < H, \Phi_0 > \Gamma^a_{aj}(p-1)\int_{\mathbb{R}^N} |1-U|^{p-2}U_j\partial_s U \\
&= -\widetilde{\mu}^{-1}\sigma^{-1} < H, \Phi_0 > \Gamma^a_{aj}\int_{\mathbb{R}^N}\partial_j U\partial_s U_0 \\
&+ \widetilde{\mu}^{-1}\sigma^{-2} < H, \Phi_0 > \Gamma^a_{aj}\int_{\mathbb{R}^N}\xi^j(|1-U|^{p-2}(1-U)-1)\partial_s U_0 \\
&- \widetilde{\mu}^{-1}\sigma^{-1} < H, \Phi_0 > \Gamma^a_{aj}(p-1)\int_{\mathbb{R}^N} |1-U|^{p-2}U_j\partial_s U.
\end{aligned}
$$

Furthemore

$$
\begin{aligned}
\int_{\mathbb{R}^N} N_1(U, w_1)\partial_s U d\xi &= -\widetilde{\mu}^{-1}\Gamma^a_{aj}\int_{\mathbb{R}^N}\partial_j w_{1,2}\partial_s U \\
&+ \widetilde{\mu}^{-1}(p-1)\sigma^{-1}H^j\int_{\mathbb{R}^N} |1-U|^{p-2}\xi^j w_{1,2}\partial_s U \\
&+ \widetilde{\mu}^{-1}(p-1)\sigma^{-1}H^j\,\Phi_0^j\int_{\mathbb{R}^N} |1-U|^{p-2}w_{1,1}\partial_s U \\
&= -\widetilde{\mu}^{-1}\sigma^{-1} < H, \Phi_0 > \Gamma^a_{aj}\int_{\mathbb{R}^N}\partial_j U_0\partial_s U \\
&+ \widetilde{\mu}^{-1}(p-1)\sigma^{-2}H^j < H\,\Phi_0 > \int_{\mathbb{R}^N} |1-U|^{p-2}\xi^j U_0\partial_s U \\
&+ \widetilde{\mu}^{-1}\sigma^{-1}(p-1) < H\,\Phi_0 > \Gamma^a_{aj}\int_{\mathbb{R}^N} |1-U|^{p-2}U_j\partial_s U.
\end{aligned}
$$

Summarizing, we get

$$
\begin{aligned}
\int_{\mathbb{R}^N} N_0(w_1) + N_1(U, w_1)d\xi = -2\widetilde{\mu}^{-1}\Gamma^a_{aj}\sigma^{-1} < H, \Phi_0 > \int_{\mathbb{R}^N}\partial_j U_0\partial_s U \\
+ \widetilde{\mu}^{-1}\sigma^{-2} < H, \Phi_0 > \Gamma^a_{aj}\int_{\mathbb{R}^N}\xi^j(|1-U|^{p-2}(1-U)-1)\partial_s U_0 \\
+ \widetilde{\mu}^{-1}(p-1)\sigma^{-2}H^j < H\,\Phi_0 > \int_{\mathbb{R}^N} |1-U|^{p-2}\xi^j U_0\partial_s U.
\end{aligned}
$$

Recall the expression of U_0 given by

$$
U_0 = \frac{1-p}{2Np}\,\xi\cdot\nabla U \; - \frac{1}{Np}U
$$

we easily have

$$
\begin{aligned}
\int_{\mathbb{R}^N}\partial_j U_0\partial_s U d\xi &= \int_{\mathbb{R}^N}\partial_s U\,\partial_j(\frac{1-p}{2Np}\xi.\nabla U \; - \frac{1}{Np}U) \\
&= -\frac{1}{Np}\int_{\mathbb{R}^N}|\nabla U|^2 + \frac{1-p}{2Np}\int_{\mathbb{R}^N}|\nabla U|^2 + \frac{1-p}{2Np}\int_{\mathbb{R}^N}\partial_s U(\xi.\nabla(\frac{\partial U}{\partial \xi^j})) \\
&= (-\frac{1}{Np} + \frac{1-p}{2Np} + \frac{1-p}{2Np}(-\frac{N}{2}))\int_{\mathbb{R}^N}|\nabla U|^2 \\
&= -\frac{1}{2N}\sigma\int_{\mathbb{R}^N}|\nabla U|^2.
\end{aligned}
$$

Integrating by parts we get

$$\begin{aligned}\int_{\mathbb{R}^N}\Big(|U-1|^{p-2}(U-1)+1\Big)<\xi,\nabla U_0>)d\xi &= -(p-1)\int_{\mathbb{R}^N}|U-1|^{p-2}U_0<\xi,\nabla U>\,d\xi\\ &- N\int_{\mathbb{R}^N}(|U-1|^{p-2}(U-1)+1)U_0\,d\xi.\end{aligned}$$

Therefore, from Formula (3.21) we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N}\Big(|U-1|^{p-2}(U-1)+1\Big)<\xi,\nabla U_0>)d\xi+(p-1)\int_{\mathbb{R}^N}|U-1|^{p-2}U_0<\xi,\nabla U>\,d\xi\\ = &-N\int_{\mathbb{R}^N}(|U-1|^{p-2}(U-1)+1)U_0\,d\xi.\\ = &\ \frac{1}{p}\int_{\mathbb{R}^N}(|U-1|^{p-2}(U-1)+1)U\,d\xi+\frac{(1-p)\sigma}{2p}\int_{\mathbb{R}^N}|\nabla U|^2d\xi.\end{aligned}$$

Putting the above formulas together we get

$$\begin{aligned}A &= \int_{\mathbb{R}^N}(N_0(w_1)+N_1(w_1,U))\partial_sU\,d\xi\\ &= -2\widetilde{\mu}^{-1}\Gamma^a_{as}\sigma^{-1}<H,\Phi_0>\Big(-\frac{1}{2N}\sigma\int_{\mathbb{R}^N}|\nabla U|^2d\xi\Big)\\ &+ \widetilde{\mu}^{-1}\Gamma^a_{as}\sigma^{-2}<H,\Phi_0>\Big[\frac{1}{p}\int_{\mathbb{R}^N}(|U-1|^{p-2}(U-1)+1)U\,d\xi+\frac{(1-p)\sigma}{2p}\int_{\mathbb{R}^N}|\nabla U|^2d\xi\Big]\\ &= \frac{1}{N}\widetilde{\mu}^{-1}\Gamma^a_{as}<H,\Phi_0>\int_{\mathbb{R}^N}|\nabla U|^2d\xi+\frac{1}{p}\widetilde{\mu}^{-1}\Gamma^a_{as}\sigma^{-2}<H,\Phi_0>\int_{\mathbb{R}^N}(|U-1|^{p-2}(U-1)+1)U\,d\xi\\ &+ \frac{(1-p)}{2p}\widetilde{\mu}^{-1}\sigma^{-1}\Gamma^a_{aj}<H,\Phi_0>\int_{\mathbb{R}^N}|\nabla U|^2d\xi.\end{aligned}$$

On the other hand, by direct computations yield

$$\begin{aligned}\int_{\mathbb{R}^N}\partial_jU\,\partial_sU &= \delta_{js}\int_{\mathbb{R}^N}|\partial_1U|^2,\\ \int_{\mathbb{R}^N}\partial^2_{kj}U\,\bar{\xi}^k\partial_sU &= \frac{1}{2}\,\delta_{js}\int_{\mathbb{R}^N}\bar{\xi}^k\partial_k(\partial_jU)^2=-\frac{N}{2}\,\delta_{js}\int_{\mathbb{R}^N}|\partial_1U|^2,\\ \Gamma^c_{ak}\Gamma^a_{cj}\,\Phi^k_0\int_{\mathbb{R}^N}\partial_jU\,\partial_sU &= \Gamma^c_{ak}\Gamma^a_{cs}\,\Phi^k_0\int_{\mathbb{R}^N}|\partial_1U|^2.\end{aligned}$$

Summarizing, we have

$$\begin{aligned}\int_{\mathbb{R}^N}S_2(\Phi_0)\,\partial_sU = -\widetilde{\mu}^{-1}\Big\{&\Delta_K\Phi^s_0+\Gamma^c_{ak}\Gamma^a_{cs}\,\Phi^k_0+(2-N)\widetilde{\mu}^{-1}\nabla_K\widetilde{\mu}\cdot\nabla_K\Phi^s_0\\ &+\frac{(p-1)}{p}\sigma^{-1}H^sH^j\Phi^j_0+2\widetilde{h}^{-1}\nabla_K\widetilde{h}\cdot\nabla_K\Phi^s_0-\psi^{-1}\sigma\,\partial^2_{sj}(\psi)\Phi^j_0\Big\}\int_{\mathbb{R}^N}|\nabla U|^2\\ &+\frac{p-1}{p}\widetilde{\mu}^{-1}\sigma^{-2}H^sH^j\Phi^j_0\int_{\mathbb{R}^N}(|1-U|^{p-2}-1)<\xi\,,\,\nabla U>d\xi\\ &+\Big(\frac{1}{N}\widetilde{\mu}^{-1}\Gamma^a_{as}+\frac{(p-1)}{2p}\widetilde{\mu}^{-1}\sigma^{-1}H^s\Big)<H,\Phi_0>\int_{\mathbb{R}^N}|\nabla U|^2d\xi\\ &+\frac{1}{p}\widetilde{\mu}^{-1}H^s\,\sigma^{-2}<H,\Phi_0>\int_{\mathbb{R}^N}(|U-1|^{p-2}(U-1)+1)U\,d\xi.\end{aligned}$$

Now, using the fact that

$$\widetilde{\mu}^{-1}\,\nabla_K\widetilde{\mu}=\frac{p-1}{2p}\,\psi^{-1}\,\nabla_K\psi\quad\text{and}\quad\widetilde{h}^{-1}\,\nabla_K\widetilde{h}=\frac{1}{p}\,\psi^{-1}\,\nabla_K\psi.$$

we obtain (recalling the definition of σ) that

$$(2-N)\widetilde{\mu}^{-1}\nabla_K\widetilde{\mu}\cdot\nabla_K\Phi_0^s+2\widetilde{h}^{-1}\nabla_K\widetilde{h}\cdot\nabla_K\Phi_0^s=\sigma\,\psi^{-1}\,\nabla_K\psi\cdot\nabla_K\Phi_0^s.$$

Hence we summarize

$$\int_{\mathbb{R}^N}S_2(\Phi_0)\,\partial_sU=-\widetilde{\mu}^{-1}\Big\{\Delta_K\Phi_0^s+\Gamma_{ak}^c\Gamma_{cs}^a\,\Phi_0^k+\sigma\,\psi^{-1}\,\nabla_K\psi\cdot\nabla_K\Phi_0^s$$
$$+\frac{(p-1)}{p}\sigma^{-1}H^s<H,\Phi_0>-\psi^{-1}\sigma\,\partial_{sj}^2(\psi)\Phi_0^j\Big\}\int_{\mathbb{R}^N}|\nabla U|^2\,d\xi$$
$$+\Big(\frac{p-1}{2p}\,\sigma^{-1}-\frac{1}{N}\Big)\widetilde{\mu}^{-1}H^s<H,\Phi_0>\int_{\mathbb{R}^N}|\nabla U|^2d\xi+\mathcal{I}$$

where $\mathcal{I}$ is explicitly given by

$$\mathcal{I}:=\frac{p-1}{p}\widetilde{\mu}^{-1}\sigma^{-2}H^s<H,\Phi_0>\int_{\mathbb{R}^N}(|1-U|^{p-2}-1)<\xi\,,\nabla U>d\xi$$
$$+\frac{1}{p}\widetilde{\mu}^{-1}\sigma^{-2}\,H^s\;<H,\Phi_0>\int_{\mathbb{R}^N}(|U-1|^{p-2}(U-1)+1)U\,d\xi$$

A straightforward computations imply that

$$\mathcal{I}=\frac{1}{p}\widetilde{\mu}^{-1}\sigma^{-2}H^s<H,\Phi_0>\Big[(p-1)\int_{\mathbb{R}^N}(|1-U|^{p-2}-1)<\xi\,,\nabla U>d\xi$$
$$+\int_{\mathbb{R}^N}(|U-1|^{p-2}(U-1)+1)U\,d\xi\Big]$$
$$=\frac{1}{p}\widetilde{\mu}^{-1}\sigma^{-2}H^s<H,\Phi_0>\Big[p\int_{\mathbb{R}^N}Ud\xi\Big]$$
$$=\frac{\widetilde{\mu}^{-1}\sigma^{-1}}{N}H^s<H,\Phi_0>\int_{\mathbb{R}^N}|\nabla U|^2d\xi.$$

On the other hand, recalling the expression of σ we have that

$$-\frac{(p-1)}{p}\sigma^{-1}+\frac{(p-1)}{2p}\sigma^{-1}-\frac{1}{N}+\frac{1}{N}\sigma^{-1}=-\frac{1}{Np}\,\sigma^{-1}.$$

We deduce that

$$\int_{\mathbb{R}^N}S_2(\Phi_0)\,\partial_sU=-\widetilde{\mu}^{-1}\Big\{\Delta_K\Phi_0^s+\Gamma_{ak}^c\Gamma_{cs}^a\,\Phi_0^k+\sigma\,\psi^{-1}\,\nabla_K\psi\cdot\nabla_K\Phi_0^s$$
$$+\frac{\sigma^{-1}}{Np}\,H^sH^j\Phi_0^j-\psi^{-1}\sigma\,\partial_{sj}^2(\psi)\Phi_0^j\Big\}\int_{\mathbb{R}^N}|\nabla U|^2\,d\xi.$$

Let now, $\mathcal{J}_K:NK\mapsto NK$ be a linear operator from the family of smooth sections of the normal bundle, NK, of K into itself, whose components are given by

$$(\mathcal{J}_K\Phi_0)^s=\Delta_K\Phi_0^s+\Gamma_{ak}^c\Gamma_{cs}^a\,\Phi_0^k+\frac{\sigma}{\psi}\,\nabla_K\psi\cdot\nabla_K\Phi_0^s+\frac{\sigma^{-1}}{Np}\,H^sH^j\Phi_0^j-\frac{\sigma}{\psi}\,\partial_{sj}^2(\psi)\Phi_0^j.\tag{3.31}$$

It then follows that

$$\int_{\mathbb{R}^N}S_2(\Phi_0)\,\partial_sU=-\widetilde{\mu}^{-1}\,(\mathcal{J}_K\Phi_0)^s(\varepsilon y)\int_{\mathbb{R}^N}|\nabla U|^2.\tag{3.32}$$

On the other hand, it is easy to check that

$$\int_{\mathbb{R}^N}S_{2,0}\,\partial_sU=0=\int_{\mathbb{R}^N}N_2(\Phi_0)\,\partial_sU.\tag{3.33}$$

Moreover, by the stationary condition (3.22) and recalling the expression of w_{11} in (3.25) we have that

$$\begin{aligned}\int_{\mathbb{R}^N} \mathfrak{G}_2\, \partial_s U &= \Big\{ -\tilde{\mu}^{-1}\Gamma^b_{bs}\int_{\mathbb{R}^N} \partial_s Z\, \partial_s U + \tilde{\mu}^{-1}\psi^{-1}(1-p)\partial_s\psi \int_{\mathbb{R}^N} |1-U|^{p-2}\bar{\xi}^s\, Z\, \partial_s U \\ &\qquad + p(p-1)\int_{\mathbb{R}^N} |U-1|^{p-2} w_{1,1} Z\, \partial_s U\Big\} e \\ &= -\tilde{\mu}^{-1}\Gamma^b_{bs} e \int_{\mathbb{R}^N} \Big\{\partial_s Z + \sigma^{-1}(1-p)|1-U|^{p-2} Z\, \bar{\xi}^s - p(p-1)|1-U|^{p-2} Z\, U_s\Big\}\partial_s U \\ &= c_0 \tilde{\mu}^{-1}\Gamma^b_{bs} e.\end{aligned}$$

Collecting the above formulas together, we see that the solvability of equation on w_2 is equivalent to the solvability of following equation on Φ_0

$$\mathcal{J}_K \Phi_0 = \mathfrak{H}_2(\bar{y}; e), \tag{3.34}$$

where $\mathfrak{H}_2(\bar{y}; e) = c_0 H e$ is a smooth bounded function. This equation is solvable by our non-degeneracy condition on K. Moreover, it is easy to check that $\Phi_0 = \Phi_0(\bar{y}; e)$ is a smooth bounded function on $\bar{y}$ and is Lipschitz continuous with respect to e.

We now go back to the equation of $w_{2,1}$

$$L_0 w_{2,1} = -\tilde{\mathfrak{F}}_2 - \mathfrak{G}_2.$$

Since $\tilde{\mathfrak{F}}_2$ and $\mathfrak{G}_2$ are smooth bounded functions of $(\varepsilon y, \bar{\xi})$, then $w_{2,1} = w_{2,1}(\varepsilon y, \bar{\xi})$ is also a smooth bounded function of $(\varepsilon y, \bar{\xi})$. Moreover, $w_{2,1} = w_{2,1}(\varepsilon y, \bar{\xi}; e)$ is Lipschitz continuous with respect to e.

3.2.3. *Higher order approximations.* To solve the equation up to an error of order ε^{j+1} for some $j \geq 3$, we use an iterative scheme of Picard's type and we argue as in the previous steps. We assume that all the functions w_i's $(1 \leq i \leq j-1)$ are constructed. The function w_j will be chosen to solve an equation similar to that of w_2 (with obvious modifications) by solving an equation of Φ_{j-2} similar to that of Φ_0. More precisely, we have

$$\begin{aligned}\tilde{S}_\varepsilon(v) &= \varepsilon^j\Big(\Delta_{\mathbb{R}^N} w_j + |U-1|^{p-2}(U-1)w_j\Big) + \varepsilon(\varepsilon^2\tilde{\mu}^{-2}\Delta_K e - \lambda_0 e)Z \\ &\quad + \varepsilon^j \mathfrak{F}_j + \varepsilon^j \mathfrak{E}_j\, e\, Z + \varepsilon^j \mathcal{A}^i_j(\varepsilon y, \bar{\xi}; \Phi_0, \cdots, \Phi_{j-3})\, e\, \partial_i Z \\ &\quad + \varepsilon^j \mathcal{B}^{i\ell}_j(\varepsilon y, \bar{\xi}; \Phi_0, \cdots, \Phi_{j-3})\, e\, \partial^2_{i\ell} Z + \varepsilon^j \mathcal{C}^i_j(\varepsilon y, \bar{\xi}; \Phi_0, \cdots, \Phi_{j-3}) \cdot \nabla_K e\, \partial_i Z \\ &\quad + \varepsilon^j \mathcal{D}^{ab}_j(\varepsilon y, \bar{\xi}; \Phi_0, \cdots, \Phi_{j-3})\, \partial^2_{ab} e\, Z + \mathcal{O}(\varepsilon^{j+1}),\end{aligned}$$

with

$$\begin{aligned}\mathfrak{F}_j &= -\tilde{\mu}^{-1}\Gamma^b_{bl}\, \partial_l w_{j-1} - \tilde{\mu}^{-1}\Gamma^c_{ak}\Gamma^a_{cs}\, \Phi^k_{j-2}\, \partial_s U - \tilde{\mu}^{-2}\Big(2\,\nabla_K\tilde{\mu}\cdot\nabla_K\Phi^s_{j-2} + \tilde{\mu}\,\Delta_K\Phi^s_{j-2}\Big)\partial_s U \\ &\quad - 2(\tilde{h}\tilde{\mu})^{-1}\Big(\nabla_K\tilde{h}\cdot\nabla_K\Phi^s_{j-2}\Big)\partial_s U - 2\tilde{\mu}^{-2}\Big(\nabla_K\tilde{\mu}\cdot\nabla_K\Phi^s_{j-2}\Big)(\bar{\xi}^i\,\partial^2_{is}U) \\ &\quad + \psi^{-1}\langle\nabla^N\psi, \Phi_0\rangle\big((1-p)|U-1|^{p-2}\big)w_{j-1} + \psi^{-1}\langle\nabla^N\psi, \Phi_{j-2}\rangle\big((1-p)|U-1|^{p-2}\big)w_1 \\ &\quad + \psi^{-1}\langle\nabla^N\psi, \Phi_{j-1}\rangle\big(|U-1|^{p-2}(1-U)-1\big) + \psi^{-1}\langle\nabla^N\psi, \frac{\bar{\xi}}{\tilde{\mu}}\rangle\big((1-p)|U-1|^{p-2}\big)\, w_{j-1} \\ &\quad + p\tilde{\mu}^{-1}\tilde{h}^{-1}\partial^2_{kl}(\psi^{\frac{1}{p}})(\varepsilon y, 0)\, \Phi^l_{j-2}\, \bar{\xi}^k\, \big(|U-1|^{p-2}(1-U)-1\big) \\ &\quad + p(p-1)|U-1|^{p-2} w_1 w_{j-1} + G_j(\varepsilon y, \bar{\xi}; w_0, \cdots, w_{j-2}, \Phi_0, \cdots, \Phi_{j-3}) \\ &= \psi^{-1}\langle\nabla^N\psi, \Phi_{j-1}\rangle\Big(|1-U|^{p-2}(1-U)-1\Big) + \tilde{\mathfrak{F}}_j\end{aligned}$$

and

$$\mathfrak{E}_j = p(p-1)|U-1|^{p-2} w_{j-1} + \psi^{-1}\langle \nabla^N \psi, \Phi_{j-2}\rangle + \widetilde{\mathfrak{E}}_j(\varepsilon y, \bar{\xi}; \Phi_0, \cdots, \Phi_{j-3}),$$

where $\mathcal{A}^i_j$, $\mathcal{B}^{i\ell}_j$, $\mathcal{C}^i_j$, $\mathcal{D}^{ab}_j$ and $\widetilde{\mathfrak{E}}_j$ are smooth bounded functions on their variables.

Except for $\varepsilon\big(-\varepsilon^2\widetilde{\mu}^{-2}\Delta_K e + \lambda_0 e\big)Z$, the term of order ε^j vanishes if and only if w_j satisfies the equation

$$\begin{aligned} L_0 w_j = & -\mathfrak{F}_j - \mathfrak{E}_j\, e\, Z - \mathcal{A}^i_j(\varepsilon y, \bar{\xi}; \Phi_0, \cdots, \Phi_{j-3})\, e\, \partial_i Z - \mathcal{B}^{i\ell}_j(\varepsilon y, \bar{\xi}; \Phi_0, \cdots, \Phi_{j-3})\, e\, \partial^2_{i\ell} Z \\ & - \mathcal{C}^i_j(\varepsilon y, \bar{\xi}; \Phi_0, \cdots, \Phi_{j-3}) \cdot \nabla_K e\, \partial_i Z - \mathcal{D}^{ab}_j(\varepsilon y, \bar{\xi}; \Phi_0, \cdots, \Phi_{j-3})\, \partial^2_{ab} e\, Z. \end{aligned}$$

As for the previous steps, by Freedholm alternative the above equation is solvable if and only if the right hand side is L^2 orthogonal to the kernel of linearized operator L_0. Recall first that, similar argument as for w_1 and w_2 yields that

$$w_{j-1} = w_{j-1,1} + \sigma^{-1}\langle H, \Phi_{j-2}\rangle U_0,$$

where $w_{j-1,1} \perp \partial_i U$ is a function which does not involve Φ_{j-2}. So, we look for a solution w_j of the form

$$w_j = w_{j,1} + \sigma^{-1}\langle H, \Phi_{j-1}\rangle U_0,$$

with $w_{j,1} \perp \partial_i U$ solves

$$\begin{aligned} L_0 w_{j,1} = & -\widetilde{\mathfrak{F}}_j - \mathfrak{E}_j\, e\, Z - \mathcal{A}^i_j(\varepsilon y, \bar{\xi}; \Phi_0, \cdots, \Phi_{j-3})\, e\, \partial_i Z - \mathcal{B}^{i\ell}_j(\varepsilon y, \bar{\xi}; \Phi_0, \cdots, \Phi_{j-3})\, e\, \partial^2_{i\ell} Z \\ & - \mathcal{C}^i_j(\varepsilon y, \bar{\xi}; \Phi_0, \cdots, \Phi_{j-3}) \cdot \nabla_K e\, \partial_i Z - \mathcal{D}^{ab}_j(\varepsilon y, \bar{\xi}; \Phi_0, \cdots, \Phi_{j-3})\, \partial^2_{ab} e\, Z. \end{aligned}$$

Since $j \geq 3$, we can write

$$\widetilde{\mathfrak{F}}_j = \widetilde{\mathfrak{F}}_j(\Phi_{j-2}) = S_{j,0} + S_j(\Phi_{j-2}),$$

where $S_{j,0} = S_{j,0}(\varepsilon y, \bar{\xi}; \Phi_0, \cdots, \Phi_{j-3})$ does not involve Φ_{j-2}, and $S_j(\Phi_{j-2})$ is the sum of linear terms of Φ_{j-2}. Since

$$\int_{\mathbb{R}^N} S_j(\Phi_{j-2})\, \partial_s U = -\widetilde{\mu}^{-1}\Big(\int_{\mathbb{R}^N} |\partial_1 U|^2\Big)(\mathcal{J}_K \Phi_{j-2})^s(\varepsilon y), \tag{3.35}$$

the equation on $w_{j,1}$ (and then on w_j) is solvable if and only if Φ_{j-2} satisfies an equation of the form

$$\mathcal{J}_K \Phi_{j-2} = \mathfrak{H}_j(\bar{y}; \Phi_0, \cdots, \Phi_{j-3}, e).$$

This latter equation is solvable by the non-degeneracy condition on K. Moreover, for any given e, by induction method one can get $\Phi_{j-2} = \Phi_{j-2}(\bar{y}; e)$ is a smooth bounded function on $\bar{y}$ and is Lipschitz continuous with respect to e. When this is done, since the right hand side of equation of $w_{j,1}$ is a smooth bounded function of $(\varepsilon y, \bar{\xi})$, we see at once that $w_{j,1} = w_{j,1}(\varepsilon y, \bar{\xi})$ is a smooth bounded function of $(\varepsilon y, \bar{\xi})$. Furthermore, $w_{j,1} = w_{j,1}(\varepsilon y, \bar{\xi}; e)$ is Lipschitz continuous with respect to e.

As a summary, given any positive integer $I \geq 3$ and let v_I be the local approximate solution constructed above. Namely,

$$v_I(y, \bar{\xi}) = U(\bar{\xi}) + \sum_{\ell=1}^{I} \varepsilon^\ell w_\ell(\varepsilon y, \bar{\xi}) + \varepsilon e(\varepsilon y) Z(\bar{\xi}). \tag{3.36}$$

We have that

$$\begin{aligned} \widetilde{S}_\varepsilon(v_I) = & \ \varepsilon\big(-\varepsilon^2\widetilde{\mu}^{-2}\Delta_K e + \lambda_0 e\big)Z + \varepsilon^{I+1}\widetilde{\mathfrak{F}}_{I+1} + \varepsilon^{I+1}\mathfrak{E}_{I+1}\, e\, Z \\ & + \varepsilon^{I+1}\mathcal{A}^i_{I+1}(\varepsilon y, \bar{\xi}; e)\, e\, \partial_i Z + \varepsilon^{I+1}\mathcal{B}^{i\ell}_{I+1}(\varepsilon y, \bar{\xi}; e)\, e\, \partial^2_{i\ell} Z \\ & + \varepsilon^{I+1}\mathcal{C}^i_{I+1}(\varepsilon y, \bar{\xi}; e) \cdot \nabla_K e\, \partial_i Z + \varepsilon^{I+1}\mathcal{D}^{ab}_{I+1}(\varepsilon y, \bar{\xi}; e)\, \partial^2_{ab} e\, Z + \mathcal{O}(\varepsilon^{I+2}), \end{aligned} \tag{3.37}$$

where

$$\begin{aligned}\mathfrak{F}_{I+1} =& -\widetilde{\mu}^{-1}\Gamma^b_{bl}\,\partial_l w_I - \widetilde{\mu}^{-1}\Gamma^c_{ak}\Gamma^a_{cs}\,\Phi^k_{I-1}\,\partial_s U - \widetilde{\mu}^{-2}\Big(2\,\nabla_K\mu\cdot\nabla_K\Phi^s_{I-1} + \widetilde{\mu}\,\Delta_K\Phi^s_{I-1}\Big)\partial_s U\\ &- 2(\widetilde{h\mu})^{-1}\Big(\nabla_K h\cdot\nabla_K\Phi^s_{I-1}\Big)\partial_s U - 2\widetilde{\mu}^{-2}\Big(\nabla_K\widetilde{\mu}\cdot\nabla_K\Phi^s_{I-1}\Big)(\bar{\xi}^i\,\partial^2_{is}U)\\ &+ \psi^{-1}\langle\nabla^N\psi,\Phi_0\rangle\big((1-p)|U-1|^{p-2}\big)w_I + \psi^{-1}\langle\nabla^N\psi,\Phi_{I-1}\rangle\big((1-p)|U-1|^{p-2}\big)w_1\\ &+ \psi^{-1}\langle\nabla^N\psi,\frac{\bar{\xi}}{\widetilde{\mu}}\rangle\big((1-p)|U-1|^{p-2}\big)\,w_I + p(p-1)|U-1|^{p-2}w_1 w_I\\ &+ p\widetilde{\mu}^{-1}\widetilde{h}^{-1}\partial^2_{kl}(\psi^{\frac{1}{p}})(\varepsilon y,0)\,\Phi^l_{I-1}\,\bar{\xi}^k\,\big(|U-1|^{p-2}(1-U)-1\big) + G_{I+1}(\varepsilon y,\bar{\xi};\Phi_0,\cdots,\Phi_{j-3})\end{aligned}$$

and

$$\mathfrak{E}_{I+1} = p(p-1)|U-1|^{p-2}w_I + \psi^{-1}\langle\nabla^N\psi,\Phi_{I-1}\rangle + \widetilde{\mathfrak{E}}_j(\varepsilon y,\bar{\xi};\Phi_0,\cdots,\Phi_{j-3}),$$

and $\mathcal{A}^i_{I+1}$, $\mathcal{B}^{i\ell}_{I+1}$, $\mathcal{C}^i_{I+1}$, $\mathcal{D}^{ab}_{I+1}$, $\widetilde{\mathfrak{E}}_{I+1}$ and G_{I+1} are smooth bounded functions on their variables and are Lipschitz continuous with respect to e.

3.3. **Global approximation.** Let v_I be the local approximate solution constructed in the previous subsection and define

$$v_I(y,\xi) = h(\varepsilon y)w_I(y,\bar{\xi}),$$

in the Fermi coordinate system. Since K is compact, by the definition of Fermi coordinate, there is a constant $\delta > 0$ such that the normal coordinate x on K_ε is well defined for $|x| < 1000\delta/\varepsilon$. We define a global approximation by

$$W(z) = \eta^\varepsilon_{3\delta}(x)v_I(y,\xi) \quad \text{for } z\in\Omega_\varepsilon, \tag{3.38}$$

where $\eta^\varepsilon_{\ell\delta}(x) := \eta(\frac{\varepsilon|x|}{\ell\delta})$ and η is a nonnegative smooth cut-off function such that

$$\eta(t) = 1 \quad \text{if } |t|<1 \qquad \text{and} \qquad \eta(t) = 0 \quad \text{if } |t|>2.$$

It is easy to see that W has the concentration property as required. Note that W depends on the parameter functions Φ_{I-1} and e, thus we can write $W = W(\,\cdot\,;\Phi_{I-1},e)$ and define the configuration space of (Φ_{I-1},e) by

(3.39)

$$\Lambda := \left\{(\Phi_{I-1},e)\ \middle|\ \begin{array}{l}\|\Phi_{I-1}\|_{C^{0,\alpha}(K)} + \|\nabla\Phi_{I-1}\|_{C^{0,\alpha}(K)} + \|\nabla^2\Phi_{I-1}\|_{C^{0,\alpha}(K)} \le 1,\\ \|e\|_{C^{0,\alpha}(K)} + \varepsilon\|\nabla e\|_{C^{0,\alpha}(K)} + \varepsilon^2\|\nabla^2 e\|_{C^{0,\alpha}(K)} \le 1\end{array}\right\}.$$

Clearly, the configuration space Λ is infinite dimensional.

For $(\Phi_{I-1},e)\in\Lambda$, it is not difficult to show that for any $0<\tau<1$, there is a positive constant C (independent of ε, Φ_{I-1}, e) such that

$$|w_I(y,\bar{\xi})| \le Ce^{-\tau|\bar{\xi}|}, \quad \forall\,(y,\bar{\xi})\in K_\varepsilon\times\mathbb{R}^N. \tag{3.40}$$

4. Proof of the Theorem 1.1

To prove Theorem 1.1, we will apply a technique already employed in many papers and which is based on the so-called infinite dimensional reduction. This technique is in some sense a generalization of the classical Lyapunov-Schmidt reduction method in an infinite dimensional setting. It has been used in many constructions in PDE and geometric analysis. We present here the main ideas referring to [12, 36, 39] and some references therein.

4.1. **Infinite dimensional reduction method.** In general there are two different approaches to set-up the problem: a first one used in [12] and [39] and a second one used in [29, 31]. Here we will present an approach which is slightly different from those.

Given $(\Phi_{I-1}, e) \in \Lambda$ and let W be the global approximate solution defined in (3.38). Our aim is to prove, applying an infinite dimensional reduction method, that there exist Φ_{I-1} and e such that a small perturbation of the global approximation W give a true solution of our problem.

Let $\underline{u_\varepsilon}(\varepsilon y)$ be the function defined in Lemma 3.1 and define the following quantities

$$E := \Delta_g W + |W + \underline{u_\varepsilon}(\varepsilon y)|^p - |\underline{u_\varepsilon}(\varepsilon y)|^p,$$

$$L_\varepsilon[\Psi] := \Delta_g \Psi + p|W-1|^{p-2}(W-1)\Psi,$$

and

$$N(\Psi) := |W + \Psi - \mathbf{q}(\varepsilon y, \varepsilon x)|^p - |W - \mathbf{q}(\varepsilon y, \varepsilon x)|^p - p|W-1|^{p-2}(W-1)\Psi.$$

Clearly, $W + \Psi$ is a solution of equation (3.4) if and only if

$$L_\varepsilon[\Psi] + E + N(\Psi) = 0. \tag{4.1}$$

To solve (4.1), we use an argument that has been in several papers, see for instance [12, 15, 36, 39] and some references therein. This argument called *gluing technique* consists in looking for a solution Ψ of the form

$$\Psi := \eta_{3\delta}^\varepsilon \Psi^\sharp + \Psi^\flat,$$

where $\Psi^\flat : \Omega_\varepsilon \to \mathbb{R}$ and $\Psi^\sharp : K_\varepsilon \times \mathbb{R}^N \to \mathbb{R}$ and where η_δ is a cut-off function defined in the previous subsection. An easy computation yields

$$\begin{aligned}
&\big[-\eta_{3\delta}^\varepsilon \Delta \Psi^\sharp - \eta_{3\delta}^\varepsilon p|W-1|^{p-2}(W-1)\Psi^\sharp - \eta_\delta^\varepsilon N(\eta_{3\delta}^\varepsilon \Psi^\sharp + \Psi^\flat) - \eta_\delta^\varepsilon E - p\eta_\delta^\varepsilon |W-1|^{p-2}(W-1)\Psi^\flat\big] \\
&+ \big[-\Delta\Psi^\flat - p(1-\eta_\delta^\varepsilon)|W-1|^{p-2}(W-1)\Psi^\flat - (1-\eta_\delta^\varepsilon)N(\eta_{3\delta}^\varepsilon \Psi^\sharp + \Psi^\flat) - (1-\eta_\delta^\varepsilon)E \\
&\qquad - 2\nabla\eta_{3\delta}^\varepsilon \nabla\Psi^\sharp - \Delta\eta_{3\delta}^\varepsilon \Psi^\sharp\big] = 0.
\end{aligned}$$

Therefore, Ψ is a solution of (4.1) if the pair $(\Psi^\flat, \Psi^\sharp)$ satisfies the following coupled system:

$$\begin{cases} -\eta_{3\delta}^\varepsilon L_\varepsilon(\Psi^\sharp) &= \eta_\delta^\varepsilon \Big[N(\eta_{3\delta}^\varepsilon \Psi^\sharp + \Psi^\flat) + E + p|W-1|^{p-2}(W-1)\Psi^\flat\Big] \\ L_\varepsilon^\flat[\Psi^\flat] &= 2\nabla\eta_{3\delta}^\varepsilon \nabla\Psi^\sharp + \Delta\eta_{3\delta}^\varepsilon \Psi^\sharp + (1-\eta_\delta^\varepsilon)\Big[N(\eta_{3\delta}^\varepsilon \Psi^\sharp + \Psi^\flat) + E\Big]. \end{cases} \tag{4.2}$$

where $L_\varepsilon^\flat[\Psi^\flat]$ is the linear operator defined by

$$L_\varepsilon^\flat[\Psi^\flat] := -\Delta\Psi^\flat - p(1-\eta_\delta^\varepsilon)|W-1|^{p-2}(W-1)\Psi^\flat \quad \text{on } \Omega_\varepsilon, \tag{4.3}$$

Then, in the support of $\eta_{3\delta}^\varepsilon$, we define

$$\Psi^\sharp := h(\varepsilon y)\Psi^*(y, \bar{\xi}), \qquad \text{with} \qquad \Psi^* : K_\varepsilon \times \mathbb{R}^N \to \mathbb{R}.$$

A straightforward computations yields

$$\eta_{3\delta}^\varepsilon \Big(\Delta_g \Psi^\sharp + p|W-1|^{p-2}(W-1)\Psi^\sharp\Big) =$$

$$\eta_{3\delta}^\varepsilon h^p \Big(\Delta_{\mathbb{R}^N}\Psi^* + \mu^{-2}\Delta_{K_\varepsilon}\Psi^* + p|\eta_{3\delta}^\varepsilon v_I - 1|^{p-2}(\eta_{3\delta}^\varepsilon v_I - 1)\Psi^* + \widetilde{B}[\Psi^*]\Big).$$

where $\widetilde{B} = \mathcal{O}(\varepsilon)$ is a linear operator defined in Subsection 3.1. Now we extend $\widetilde{B}$ to $K_\varepsilon \times \mathbb{R}^N$ by defining

$$\mathbb{L}_\varepsilon[\Psi^*] := \Delta_{\mathbb{R}^N}\Psi^* + \mu^{-2}\Delta_{K_\varepsilon}\Psi^* + p|\eta_{3\delta}^\varepsilon v_I - 1|^{p-2}(\eta_{3\delta}^\varepsilon v_I - 1)\Psi^* + \eta_{6\delta}^\varepsilon \widetilde{B}[\Psi^*] \quad \text{on } K_\varepsilon \times \mathbb{R}^N,$$

and

$$L^*_\varepsilon[\Psi^*] := \Delta_{\mathbb{R}^N}\Psi^* + \mu^{-2}\Delta_{K_\varepsilon}\Psi^* + p|U-1|^{p-2}(U-1)\Psi^* = -L_0[\Psi^*] + \mu^{-2}\Delta_{K_\varepsilon}\Psi^* \quad \text{on } K_\varepsilon \times \mathbb{R}^N.$$

Since $\eta^\varepsilon_{3\delta}\cdot\eta^\varepsilon_\delta = \eta^\varepsilon_\delta$ and $\eta^\varepsilon_{3\delta}\cdot\eta^\varepsilon_{6\delta} = \eta^\varepsilon_{3\delta}$, then Ψ is a solution of (4.1) if the pair $(\Psi^\flat,\Psi^*)$ solves the following coupled system:

$$\begin{cases} L^\flat_\varepsilon[\Psi^\flat] = 2\nabla\eta^\varepsilon_{3\delta}\nabla(h\Psi^*) + \Delta\eta^\varepsilon_{3\delta}h\Psi^* + (1-\eta^\varepsilon_\delta)\Big[N(\eta^\varepsilon_{3\delta}h\Psi^* + \Psi^\flat) + E\Big],\\ L^*_\varepsilon[\Psi^*] = -\eta^\varepsilon_\delta\, h^{-p}\Big[E + N(\eta^\varepsilon_{3\delta}h\Psi^* + \Psi^\flat) + p|W-1|^{p-2}(W-1)\Psi^\flat\Big] - (\mathbb{L}_\varepsilon - L^*_\varepsilon)[\Psi^*]. \end{cases}$$

It is easy to check that

$$(\Delta_g\eta^\varepsilon_{3\delta})h\Psi^* + 2\nabla_g\eta^\varepsilon_{3\delta}\cdot\nabla_g(h\Psi^*) = (1-\eta^\varepsilon_\delta)\Big[(\Delta_g\eta^\varepsilon_{3\delta})h\Psi^* + 2\nabla_g\eta^\varepsilon_{3\delta}\cdot\nabla_g(h\Psi^*)\Big]$$

and

$$(1-\eta^\varepsilon_\delta) = (1-\eta^\varepsilon_\delta)(1-\eta^\varepsilon_{\delta/2}).$$

Now, we define

$$\begin{aligned} \mathcal{N}_\varepsilon(\Psi^\flat,\Psi^*,\Psi_{I-1},e) \;&:=\; (\Delta_g\eta^\varepsilon_{3\delta})h\Psi^* + 2\nabla_g\eta^\varepsilon_{3\delta}\cdot\nabla_g(h\Psi^*)\\ &+\; (1-\eta^\varepsilon_{\delta/2})\Big[E + N(\eta^\varepsilon_{3\delta}\Psi^\sharp + \Psi^\flat)\Big], \end{aligned}$$

and

$$\begin{aligned} \mathcal{M}_\varepsilon(\Psi^\flat,\Psi^*,\Psi_{I-1},e) \;&:=\; -\eta^\varepsilon_\delta\, h^{-p}\Big[E + N(\eta^\varepsilon_{3\delta}h\Psi^* + \Psi^\flat) + p|W-1|^{p-2}(W-1)\Psi^\flat\Big]\\ &-\; (\mathbb{L}_\varepsilon - L^*_\varepsilon)[\Psi^*]. \end{aligned}$$

With this definitions in mind we conclude that $W+\Psi$ is a solution of equation (3.4) if $(\Psi^\flat,\Psi^*,\Phi_{I-1},e)$ solves the following system:

$$\begin{cases} L^\flat_\varepsilon[\Psi^\flat] = (1-\eta^\varepsilon_\delta)\,\mathcal{N}_\varepsilon(\Psi^\flat,\Psi^*,\Phi_{I-1},e),\\ L^*_\varepsilon[\Psi^*] = \mathcal{M}_\varepsilon(\Psi^\flat,\Psi^*,\Phi_{I-1},e). \end{cases} \tag{4.4}$$

To solve the above system (4.4), we first study the linear theory : we prove the invertibility of the operator $L^\flat_\varepsilon$ namely we have the solvability of equation $L^\flat_\varepsilon[\Psi^\flat] = \Psi$. On the other hand, one can check at once that L^*_ε has bounded kernels, e.g., $\partial_j U$, $j = 1,\dots,N$. Actually, since L_0 has a negative eigenvalue λ_0 with the corresponding eigenfunction Z, there may be more bounded kernels of L^*_ε.

Let Ψ be a function defined on $K_\varepsilon\times\mathbb{R}^N$, we define Π to be the $L^2(d\bar\xi)$-orthogonal projection on $\partial_j U$'s and Z, namely

$$\Pi[\Psi] := \Big(\Pi_1[\Psi],\dots,\Pi_N[\Psi],\Pi_{N+1}[\Psi]\Big), \tag{4.5}$$

where for $j = 1,\dots,N$,

$$\Pi_j[\Psi] := \frac{1}{c_0}\int_{\mathbb{R}^N}\Psi(y,\bar\xi)\,\partial_j U(\bar\xi)\,d\bar\xi, \quad \text{with } c_0 = \int_{\mathbb{R}^N}|\partial_1 U|^2\,d\bar\xi,$$

and

$$\Pi_{N+1}[\Psi] := \int_{\mathbb{R}^N}\Psi(y,\bar\xi)\,Z(\bar\xi)\,d\bar\xi.$$

Let us also denote by $\Pi^{\perp}$ the orthogonal projection on the orthogonal of $\partial_j U$'s and Z, namely

$$\Pi^{\perp}[\Psi] := \Psi - \sum_{j=1}^{N} \Pi_j[\Psi]\, \partial_j U - \Pi_{N+1}[\Psi]\, Z.$$

With these notations, as in the Lyapunov-Schmidt reduction, solving the system (4.4) amounts to solving the system

$$\begin{cases} L_\varepsilon^\flat[\Psi^\flat] = (1-\eta_\delta^\varepsilon)\mathcal{N}_\varepsilon(\Psi^\flat, \Psi^*, \Phi_{I-1}, e), \\ L_\varepsilon^*[\Psi^*] = \Pi^\perp\Big[\mathcal{M}_\varepsilon(\Psi^\flat, \Psi^*, \Phi_{I-1}, e)\Big], \\ \Pi\Big[\mathcal{M}_\varepsilon(\Psi^\flat, \Psi^*, \Phi_{I-1}, e)\Big] = 0. \end{cases} \tag{4.6}$$

It is to see that one can write

$$E = \eta_{3\delta}^\varepsilon\, h^p\, \widetilde{S}_\varepsilon(v_I) + (\Delta_g \eta_{3\delta}^\varepsilon)(hv_I) + 2(\nabla_g \eta_{3\delta}^\varepsilon)\cdot\nabla_g(hv_I) + \eta_{3\delta}^\varepsilon\Big[|u_\varepsilon(\varepsilon y)|^p - |hv_I - u_\varepsilon(\varepsilon y)|^p\Big].$$

Hence by (3.37),

$$\begin{aligned} \mathcal{M}_\varepsilon(\Psi^\flat, \Psi^*, \Phi_{I-1}, e) =& \varepsilon(\varepsilon^2\mu^{-2}\Delta_K e - \lambda_0 e) Z + \varepsilon^{I+1} S_{I+1}(\Phi_{I-1}) \\ &+ \varepsilon^{I+1} G_{I+1}(\varepsilon y, \bar{\xi}; e) + \varepsilon^{I+2} J_{I+1}(\varepsilon y, \bar{\xi}; \Phi_{I-1}, e) \\ &- \eta_\delta^\varepsilon\, h^{-p}\Big[N(\eta_{3\delta}^\varepsilon h\Psi^* + \Psi^\flat) + p|W-1|^{p-2}(W-1)\Psi^\flat\Big] - (\mathbb{L}_\varepsilon - L_\varepsilon^*)[\Psi^*]. \end{aligned}$$

On the other hand, since

$$\int_{\mathbb{R}^N} S_{I+1}(\Phi_{I-1})\, \partial_s U = c_0 \mu^{-1} (\mathcal{J}_K \Phi_{I-1})^s(\varepsilon y), \tag{4.7}$$

by some rather tedious and technical computations, one can show that

(4.8)

$$\Pi\Big[\mathcal{M}_\varepsilon(\Psi^\flat, \Psi^*, \Phi_{I-1}, e)\Big] = 0 \Longleftrightarrow \begin{cases} \varepsilon^{I+1}\mathcal{J}_K[\Phi_{I-1}] = \varepsilon^{I+1}\mathfrak{H}_{I+1}(\bar{y}; e) + \mathcal{M}_{\varepsilon,1}(\Psi\flat, \Psi^*, \Phi_{I-1}, e); \\ \varepsilon\, \mathcal{K}_\varepsilon[e] = \mathcal{M}_{\varepsilon,2}(\Psi^\flat, \Psi^*, \Phi_{I-1}, e), \end{cases}$$

where $\mathfrak{H}_{I+1}(\bar{y}; e)$ is a smooth bounded function on $\bar{y}$ and is Lipschitz continuous with respect to e, $\mathcal{J}_K$ is the Jacobi operator on K, and $\mathcal{K}_\varepsilon$ is a Schrödinger operator defined by

$$\mathcal{K}_\varepsilon[e] := \varepsilon^2 \Delta_K e - \lambda_0 \mu^2 e \tag{4.9}$$

where λ_0 is the unique negative eigenvalue of L_0.

We summarize the above discussion by saying that the function

$$u = W(\,\cdot\,; \Phi_{I-1}, e) + \eta_{3\delta}^\varepsilon\, h\, \Psi^* + \Psi^\flat,$$

is a solution of the equation

$$\Delta_g u + |u - \mathbf{q}(\varepsilon z)|^p - |\mathbf{q}(\varepsilon z)|^p = 0,$$

if the functions $\Psi^\flat$, Ψ^*, Φ_{I-1} and e satisfy the following system

$$\begin{cases} L_\varepsilon^\flat[\Psi^\flat] = (1-\eta_\delta^\varepsilon)\, \mathcal{N}_\varepsilon(\Psi^\flat, \Psi^*, \Phi_{I-1}, e), \\ L_\varepsilon^*[\Psi^*] = \Pi^\perp\Big[\mathcal{M}_\varepsilon(\Psi^\flat, \Psi^*, \Phi_{I-1}, e)\Big], \\ \varepsilon^{I+1}\mathcal{J}_K[\Phi_{I-1}] = \varepsilon^{I+1}\mathfrak{H}_{I+1}(\bar{y}; e) + \mathcal{M}_{\varepsilon,1}(\Psi^\flat, \Psi^*, \Phi_{I-1}, e), \\ \varepsilon\, \mathcal{K}_\varepsilon[e] = \mathcal{M}_{\varepsilon,2}(\Psi^\flat, \Psi^*, \Phi_{I-1}, e). \end{cases} \tag{4.10}$$

Once the system (4.10) is solved, one can prove the positivity of u by contradiction since both $\Psi^\flat$ and Ψ^* are small.

4.2. **Mapping properties of the linear operators.** The main aim of this subsection is to solve system (4.10) using a fixed point theorem. The first main ingredient in doing this is to develop a linear theory for the linear operators appearing in system (4.10) . We will study them one by one.

4.2.1. *Analysing the linear operator $L_\varepsilon^\flat$.* To deal with the term $-\eta_\delta^\varepsilon ph^{-p}|W-1|^{p-2}(W-1)\Psi^\flat$ in $\mathcal{M}_\varepsilon(\Psi^\flat, \Psi^*, \Phi_{I-1}, e)$ in applying a fixed point theorem, one needs to choose norms with the property that $\mathcal{M}_\varepsilon(\Psi^\flat, \Psi^*, \Phi_{I-1}, e)$ depends slowly on $\Psi^\flat$. To this end, we define

$$\|\Psi^\flat\|_{\varepsilon,\infty} = \|(1-\eta_{\delta/4}^\varepsilon)\Psi^\flat\|_\infty + \frac{1}{\varepsilon}\|\eta_{\delta/4}^\varepsilon \Psi^\flat\|_\infty. \tag{4.11}$$

With this notation, by the exponential decay of W, we have

$$\|\mathcal{M}_\varepsilon(\Psi^\flat, \Psi^*, \Phi_{I-1}, e)\|_\infty \le C\varepsilon\|\Psi^\flat\|_{\varepsilon,\infty}$$

and

$$\|\mathcal{M}_\varepsilon(\Psi_1^\flat, \Psi^*, \Phi_{I-1}, e) - \mathcal{M}_\varepsilon(\Psi_2^\flat, \Psi^*, \Phi_{I-1}, e)\|_\infty \le C\varepsilon\|\Psi_1^\flat - \Psi_2^\flat\|_{\varepsilon,\infty}.$$

We have the following lemma.

Lemma 4.1. *For any function $\Xi(z) \in L^\infty(M_\varepsilon)$, there is a unique bounded solution Ψ of*

$$L_\varepsilon^\flat[\Psi] = (1-\eta_\delta^\varepsilon)\Xi. \tag{4.12}$$

Moreover, there exists a constant $C > 0$ (independent of ε) such that

$$\|\Psi\|_{\varepsilon,\infty} \le C\|\Xi\|_\infty. \tag{4.13}$$

For $\Psi^\flat \in C_0^{0,\alpha}(M_\varepsilon)$, we define

$$\|\Psi^\flat\|_{\varepsilon,\alpha} = \|(1-\eta_{\delta/4}^\varepsilon)\Psi^\flat\|_{C_0^{0,\alpha}} + \frac{1}{\varepsilon}\|\eta_{\delta/4}^\varepsilon \Psi^\flat\|_{C_0^{0,\alpha}}. \tag{4.14}$$

As a consequence of standard elliptic estimates, the following lemma holds.

Lemma 4.2. *For any function $\Xi \in C_0^{0,\alpha}(M_\varepsilon)$, there is a unique solution $\Psi \in C_0^{2,\alpha}(M_\varepsilon)$ of*

$$L_\varepsilon^\flat[\Psi] = (1-\eta_\delta^\varepsilon)\Xi. \tag{4.15}$$

Moreover, there exists a constant $C > 0$ (independent of ε) such that

$$\|\Psi\|_{2,\varepsilon,\alpha} := \|\Psi\|_{\varepsilon,\alpha} + \|\nabla\Psi\|_{\varepsilon,\alpha} + \|\nabla^2\Psi\|_{\varepsilon,\alpha} \le C\|\Xi\|_{C_0^{2,\alpha}(M_\varepsilon)}. \tag{4.16}$$

4.2.2. *Studying the model linear operator L_ε^*.* We will first prove an injectivity result for L_ε^*. Using this we obtain some a priori estimates and existence result for solutions of $L_\varepsilon^*[\Psi] = \Xi$ under the orthogonality conditions $\Pi[\Psi] = 0 = \Pi[\Xi]$. We have the validity of following lemma.

Lemma 4.3 (The injectivity result)**.** *Suppose that $\Psi \in L^\infty(K_\varepsilon \times \mathbb{R}^N)$ satisfies $L_\varepsilon^*[\Psi] = 0$ and $\Pi[\Psi] = 0$. Then $\Psi \equiv 0$.*

Proof. We first mention that a bounded solution Ψ of $L_\varepsilon^*[\Psi] = 0$ decays exponentially in the variables $\bar\xi$. This follows from the exponential decay of $U(\bar\xi)$ and the maximum principle. Then one can define

$$f(y) := \int_{\mathbb{R}^N} \Psi^2(y, \bar\xi)\, d\bar\xi, \quad \forall\, y \in K_\varepsilon.$$

Our aim is to prove that f is identically equal to zero. We first recall the following inequality

$$\int_{\mathbb{R}^N}\left\{|\nabla_{\bar{\xi}}\Psi|^2+p|U-1|^{p-2}(1-U)\Psi^2\right\}d\bar{\xi}\geq\gamma_0\int_{\mathbb{R}^N}\left(\Psi^2+|\nabla_{\bar{\xi}}\Psi|^2\right)d\bar{\xi} \tag{4.17}$$

with $\gamma_0>0$ independent of $\varepsilon>0$, see (4.9) in [7] for the proof. This together with the fact that $L_\varepsilon^*[\Psi]=0$ and $\Pi(\Psi)=0$ yield

$$\begin{aligned}\Delta_{K_\varepsilon}f&=\int_{\mathbb{R}^N}2\Psi\Delta_{K_\varepsilon}\Psi\,d\bar{\xi}+\int_{\mathbb{R}^N}2|\nabla_{K_\varepsilon}\Psi|^2\,d\bar{\xi}\\&=2\mu^2\int_{\mathbb{R}^N}\left\{|\nabla_{\bar{\xi}}\Psi|^2-p|U-1|^{p-2}(U-1)\Psi^2\right\}d\bar{\xi}+2\int_{\mathbb{R}^N}|\nabla_{K_\varepsilon}\Psi|^2\,d\bar{\xi}\\&\geq 2\mu^2\gamma_0\int_{\mathbb{R}^N}\left(\Psi^2(y,\bar{\xi})+|\nabla_{\bar{\xi}}\Psi(y,\bar{\xi})|^2\right)d\bar{\xi},\\&\geq 2\mu^2\gamma_0\int_{\mathbb{R}^N}\Psi^2(y,\bar{\xi})\,d\bar{\xi}=2\mu^2\gamma_0 f.\end{aligned}$$

Integrating the above inequality and using the fact that f is nonnegative and K_ε is compact, we get $f\equiv 0$. Notice that if K_ε is non compact, one can first show that f goes to zero at infinity by the comparison theorem and then get $f\equiv 0$ by the maximum principle. □

Remark 4.1. *Actually, following the argument of proof of Lemma 3.7 in [36], one can show that*

$$\Psi=\sum_{j=1}^{N}c^j(y)\,\partial_j U+c^{N+1}(y)\,Z, \tag{4.18}$$

if Ψ is a bounded solution of $L_\varepsilon^*[\Psi]=0$, where $c_j(y)$ $(j=1,\dots,N)$ can be any bounded function, but $c^{N+1}(y)$ must satisfy the equation

$$\Delta_{K_\varepsilon}c^{N+1}=\lambda_0\mu^2c^{N+1}. \tag{4.19}$$

It is worth mentionning that (4.19) is just another form of $\mathcal{K}_\varepsilon[e]=0$. When ε satisfies some gap condition (cf. Proposition 4.3 below), equation (4.19) does not have a bounded solution.

Moreover, one can show that under the orthogonal conditions $\Pi[\Psi]=0$, the linear operator L_ε^* has only negative eigenvalues λ_j^ε's and there exists a constant c_0 such that

$$\lambda_j^\varepsilon\leq -c_0<0.$$

We next prove a subjectivity result for L_ε^*. Before stating it, we define the following wighted Hölder norm (one can also use weighted Sobolev norms)

$$\|\Psi\|_{\varepsilon,\alpha,\rho}:=\sup_{(y,\bar{\xi})\in K_\varepsilon\times\mathbb{R}^N}e^{\rho|\bar{\xi}|}\|\Psi\|_{C^{0,\alpha}(B_1((y,\bar{\xi})))},$$

for *small* positive constants α and ρ. The following result hold true.

Proposition 4.1 (The surjectivity result)**.** *For any function Ξ with $\|\Xi\|_{\alpha,\sigma}<\infty$ and $\Pi[\Xi]=0$, the problem*

$$L_\varepsilon^*[\Psi]=\Xi \tag{4.20}$$

has a unique solution Ψ with $\Pi[\Psi]=0$. Moreover, the following estimate holds:

$$\|\Psi\|_{2,\varepsilon,\alpha,\rho}:=\|\Psi\|_{\varepsilon,\alpha,\rho}+\|\nabla\Psi\|_{\varepsilon,\alpha,\rho}+\|\nabla^2\Psi\|_{\varepsilon,\alpha,\rho}\leq C\|\Xi\|_{\varepsilon,\alpha,\rho}, \tag{4.21}$$

where C is a constant independent of ε.

4.2.3. *Non-degeneracy condition and invertibility of $\mathcal{J}_K$.*

Proposition 4.2. *Suppose that K is non-degenerate, then for any $\tilde{\Phi} \in (C^{0,\alpha}(K))^N \cap NK$, there exists a unique $\Phi \in (C^{2,\alpha}(K))^N \cap NK$ such that*

$$\mathcal{J}_K[\Phi] = \tilde{\Phi} \tag{4.22}$$

with the property

$$\|\Phi\|_{2,\alpha} := \|\Phi\|_{C^{0,\alpha}(K)} + \|\nabla\Phi\|_{C^{0,\alpha}(K)} + \|\nabla^2\Phi\|_{C^{0,\alpha}(K)} \leq C\|\tilde{\Phi}\|_{C^{0,\alpha}(K)}, \tag{4.23}$$

where C is a positive constant depending only on K.

Proof. Since the Jacobi operator $\mathcal{J}_K$ is self-adjoint, this result follows from the standard elliptic estimates, cf. [18, 22]. ☐

4.2.4. *Gap condition and invertibility of $\mathcal{K}_\varepsilon$.*

Proposition 4.3. *There is a sequence $\varepsilon = \varepsilon_j \searrow 0$ such that for any $\varphi \in C^{0,\alpha}(K)$, there exists a unique $e \in C^{2,\alpha}(K)$ such that*

$$\mathcal{K}_\varepsilon[e] = \varphi \tag{4.24}$$

with the property

$$\|e\|_* := \|e\|_{C^{0,\alpha}(K)} + \varepsilon\|\nabla e\|_{C^{0,\alpha}(K)} + \varepsilon^2\|\nabla^2 e\|_{C^{0,\alpha}(K)} \leq C\varepsilon^{-3k}\|\varphi\|_{C^{0,\alpha}(K)}, \tag{4.25}$$

where C is a positive constant independent of ε_j.

Proof. The proof relies of various considerations on the asymptotic behaviour of the small eigenvalues of $\mathcal{K}_\varepsilon$, and Weyl's asymptotic formula. In fact, consider the eigenvalue problem

$$\mathcal{K}_\epsilon e = \lambda e \quad \text{in } K.$$

For any $\epsilon > 0$, the eigenvalues are given by a sequence $\lambda_j(\epsilon)$, characterized by the Courant-Fisher formulas: if M_j (resp.M_{j-1}) represents the family of j dimensional (resp. $j-1$ dimensional) subspaces of $H^2(K)$, then

$$\begin{aligned} \lambda_j(\epsilon) &= \sup_{M \in M_{j-1}} \inf_{e \in M^\perp \setminus \{0\}} Q_\epsilon(e,e) \\ &= \inf_{M \in M_j} \sup_{e \in M^\perp \setminus \{0\}} Q_\epsilon(e,e). \end{aligned} \tag{4.26}$$

where we have set

$$Q_\epsilon(e,e) = \frac{\int_K \mathcal{K}_\epsilon e.e}{\int_K |e|^2}$$

and $\perp$ denotes orthogonality with respect to the L^2-scalar product. We have the following result.

Lemma 4.4. *There exits a number $\epsilon_* > 0$, such that for all $0 < \epsilon_1 < \epsilon_2 < \epsilon_*$, and all $j \geq 1$ and for some $\gamma_-, \gamma_+ > 0$, the following inequalities hold.*

$$(\epsilon_2 - \epsilon_1)\frac{\gamma_-}{2\epsilon_2^2} \leq \epsilon_2^{-1}\lambda_j(\epsilon_2) - \epsilon_1^{-1}\lambda_j(\epsilon_1) \leq 2(\epsilon_2 - \epsilon_1)\frac{\gamma_+}{\epsilon_1^2} \tag{4.27}$$

In particular, the functions $\epsilon \in (0, \epsilon_*) \longrightarrow \lambda_j(\epsilon)$ are continuous.

Proof. We prove that, for $0 < \epsilon_1 < \epsilon_2$ and e such that $\int_K e^2 = 1$, we have

$$\epsilon_1^{-1}Q_{\epsilon_1}(e,e) + (\epsilon_2 - \epsilon_1)\frac{\gamma_-}{2\epsilon_2^2} \leq \epsilon_2^{-1}Q_{\epsilon_2}(e,e) \leq \epsilon_1^{-1}Q_{\epsilon_1}(e,e) + (\epsilon_2 - \epsilon_1)\frac{2\gamma_+}{\epsilon_1^2}$$

$\gamma_-, \gamma_+ > 0$. From this and formulas (4.26) estimate (4.27) follows at once. ☐

As a consequence we get

Corollary 4.1. *There exists a number $\delta > 0$ such that for any $\epsilon_2 > 0$ and j such that*

$$\epsilon_2 + |\lambda_j(\epsilon_2)| < \delta$$

and any ϵ_1 with $\frac{1}{2}\epsilon_2 \leq \epsilon_1 < \epsilon_2$, we have that :

$$\lambda_j(\epsilon_1) < \lambda_j(\epsilon_2).$$

Proof. Let us consider small numbers $\epsilon_1 \geq \frac{\epsilon_2}{2}$, Then from (4.27) we find that:

$$\lambda_j(\epsilon_1) \geq \lambda_j(\epsilon_2) + \frac{\epsilon_1 - \epsilon_2}{\epsilon_2}[\lambda_j(\epsilon_2) + \delta\frac{\epsilon_1}{\epsilon_2}]$$

where $\delta > 0$. From here the desired result immediately follows. $\square$

Let us consider the numbers $\overline{\epsilon_l} := 2^{-l}$ for large $l \geq 1$. Define the sets

$$F_l := \Big\{\epsilon_l \in (\overline{\epsilon}_{l+1}, \overline{\epsilon}_l);\ \ker \mathcal{K}_\epsilon \neq \{0\}\Big\} \tag{4.28}$$

if $\epsilon \in F_l$ then for some j we have that, $\lambda_j(\epsilon) = 0$, It follows that $\lambda_j(\overline{\epsilon}_{l+1}) < 0$. Indeed, let us assume the opposite. Then, given $\delta > 0$, the continuity of $\epsilon \to \lambda_j(\epsilon)$ implies the existence of $\tilde{\epsilon}$ with $\frac{1}{2}\epsilon \leq \tilde{\epsilon} < \epsilon$ and $0 \leq \lambda_j(\tilde{\epsilon}) < \delta$. If δ is chosen as in the above Corollary, and l is large enough so that $2^{-l} < \delta$, we obtain a contradiction. As a consequence, for all l large enough

$$\mathrm{Card}(F_l) \leq N(\overline{\epsilon}_{l+1}) \tag{4.29}$$

where $N(\epsilon)$ denotes the number of negative eigenvalues of $\mathcal{K}_\varepsilon$. Our next task is then to estimate this number of negative eigenvalues for ϵ sufficiently small. For this aim we consider for $a_+ > 0$ such that $a_+ > \lambda_1$, the following model operator

$$\mathcal{K}_\epsilon^+ := -\Delta_K - \frac{a_+}{\epsilon}. \tag{4.30}$$

Let $\lambda_j^+(\epsilon)$ denote its eigenvalues. From the Courant-Fisher characterization we see that $\lambda_j(\epsilon) \leq \lambda_j^+(\epsilon)$ for small j and ϵ. Therefore, $N(\epsilon) \leq N^+(\epsilon)$, where $N^+(\epsilon)$ designates the number of negative eigenvalues of $\mathcal{K}_\epsilon^+$. Let us denote by μ_j the eigenvalues of $-\Delta_K$(ordinates and counted with their multiplicity). Then Weyl's asymptotic formula there exists a constant $C_K > 0$, depending only on $k = dimK$ such that

$$\mu_j = C_K j^{\frac{2}{k}} + o(j^{\frac{2}{k}}) \quad as\ j \to \infty. \tag{4.31}$$

Using the fact that $\lambda_j^+(\epsilon) = \mu_j - \frac{a_+}{\epsilon}$ and (4.31), we then find that

$$N_+(\epsilon) = C\epsilon^{-\frac{k}{2}} + o(\epsilon^{-\frac{k}{2}}) \quad quand\ \epsilon \to \infty \tag{4.32}$$

As a conclusion, using (4.29) we find

$$\mathrm{Card}(F_l) \leq N(\overline{\epsilon}_{l+1}) \leq C\overline{\epsilon}_{l+1}^{-\frac{k}{2}} \leq C2^{\frac{lk}{2}}. \tag{4.33}$$

Hence there exists an interval $(a_l, b_l) \subset (\overline{\epsilon}_{l+1}\overline{\epsilon}_l)$ such that $a_l, b_l \in F_l, Ker(\mathcal{K}_\epsilon) = \{0\}, \epsilon \in (a_l, b_l)$ and

$$b_l - a_l \geq \frac{\overline{\epsilon}_l - \overline{\epsilon}_{l+1}}{\mathrm{Card}(F_l)} \geq C\overline{\epsilon}_l^{1+\frac{k}{2}} \geq C2^{-l(1+\frac{k}{2})}. \tag{4.34}$$

Letting

$$\epsilon_l := \frac{1}{2}(b_l - a_l)$$

we claim that for $c > 0$ and all j, we have

$$|\lambda_j(\epsilon_l)| \geq c\overline{\epsilon}_l^{\frac{k}{2}}. \tag{4.35}$$

Indeed, assuming the opposite, namely that for some j we have, $|\lambda_j(\epsilon_l)| \le \delta\varepsilon_l^{\frac{k}{2}}$, where δ was chosen a priori sufficiently small. It follows that $0 < \lambda_j(\epsilon_l) < \delta\epsilon_l^{\frac{k}{2}}$. Then, from Lemma 4.4,

$$\lambda_j(a_l) \le \lambda_j(\epsilon_l) - (\epsilon_l - a_l)\frac{1}{\epsilon_l}[\lambda_j(\epsilon_l) + \gamma\frac{a_l}{2\epsilon_l}].$$

Thus, (4.34) and (4.35) require that

$$\lambda_j(a_l)\delta\epsilon_l^{\frac{k}{2}} - c\frac{\bar\epsilon_l}{2\epsilon_l}\bar\epsilon_l^{\frac{k}{2}}[\lambda_j(\epsilon_l) + \frac{\gamma a_l}{2\epsilon_l}] < 0,$$

for γ small enough. It follows that $\lambda_j(\epsilon)$ must vanish at some $\epsilon \in (a_l, b_l)$, and we have thus reached a contradiction with the choice of the interval (a_l, b_l).
The case $-\delta\epsilon_l^{\frac{k}{2}} \le \lambda_j(\epsilon_l) < 0$ is handled similarly. In that case we get $\lambda_j(b_l) > 0$.
The proof of existence and estimate (4.35) is thus complete.
The uniqueness of solutions satisfying the estimate (4.25) follows from standard elliptic estimates and Sobolev embedding theorem. □

4.3. **Nonlinear scheme and proof of Theorem 1.1.** We have the main ingredients to complete the proof of our main result Theorem 1.1. Indeed, by the analysis in the previous sections, to proof Theorem 1.1 we are reduced to prove solvability of (4.10). This can be done using a contraction mapping argument. To do this, we let $\Phi_{I-1} = \Phi_{I-1,0} + \widetilde\Phi_{I-1}$, where $\Phi_{I-1,0}$ solve the equation

$$\mathcal{J}_K[\Phi_{I-1,0}] = \mathfrak{H}_{I+1}(\bar y; e). \tag{4.36}$$

Thus $\Phi_{I-1,0} = \Phi_{I-1,0}(\bar y; e)$. Moreover, the reduced system (4.10) becomes

$$\begin{cases} L_\varepsilon^\flat[\Psi^\flat] = (1-\eta_\delta^\varepsilon)\mathcal{N}_\varepsilon(\Psi^\flat, \Psi^*, \Phi_{I-1}, e), \\ L_\varepsilon^*[\Psi^*] = \Pi^\perp\Big[\mathcal{M}_\varepsilon(\Psi^\flat, \Psi^*, \Phi_{I-1}, e)\Big], \\ \varepsilon^{I+1}\mathcal{J}_K[\widetilde\Phi_{I-1}] = \widetilde{\mathcal{M}}_{\varepsilon,1}(\Psi^\flat, \Psi^*, \widetilde\Phi_{I-1}, e), \\ \varepsilon\,\mathcal{K}_\varepsilon[e] = \widetilde{\mathcal{M}}_{\varepsilon,2}(\Psi^\flat, \Psi^*, \widetilde\Psi_{I-1}, e). \end{cases} \tag{4.37}$$

In order to prove that the above system possesses a solution we need to estimate the size of error terms and the Lipschitz continuity of the functions $\mathcal{N}_\varepsilon$, $\mathcal{M}_\varepsilon$ and $\widetilde{\mathcal{M}}_{\varepsilon,j}$ ($j = 1, 2$) with respect to their arguments. This is the purpose of the following two lemmas.

Lemma 4.5 (Size of the error)**.** *There is a constant C independent of ε such that the following estimates hold:*

$$\|\mathcal{N}_\varepsilon(0,0,0,0)\|_{C_0^{2,\alpha}(M_\varepsilon)} + \|\Pi^\perp[\mathcal{M}_\varepsilon(0,0,0,0)]\|_{\varepsilon,\alpha,\rho} \le C\varepsilon^{I+1}. \tag{4.38}$$

Moreover,

$$\|\widetilde{\mathcal{M}}_{\varepsilon,1}(0,0,0,0)\|_{C^{0,\alpha}(K)} \le C\varepsilon^{I+2}, \quad \|\widetilde{\mathcal{M}}_{\varepsilon,2}(0,0,0,0)\|_{C^{0,\alpha}(K)} \le C\varepsilon^{I+1}. \tag{4.39}$$

Proof. It follows from the definitions and the estimate (3.40). □

We next define

$$\mathcal{B}_\lambda := \Big\{(\Psi^\flat, \Psi^*, \Phi_{I-1}, e) \,\Big|\, \quad \|\Psi^\flat\|_{2,\varepsilon,\alpha} \le \lambda\varepsilon^{I+1}, \|\Psi^*\|_{2,\varepsilon,\alpha,\rho} \le \lambda\varepsilon^{I+1}, \\ \|\Phi_{I-1}\|_{2,\alpha} \le \lambda\varepsilon, \|e\|_* \le \lambda\varepsilon^{I-3k}\Big\} \tag{4.40}$$

for some positive real number λ. We get

Lemma 4.6 (Lipschitz continuity). *Given* $(\Psi_1^\flat, \Psi_1^*, \Phi_{I-1}, e_1), (\Psi_2^\flat, \Psi_2^*, \widetilde{\Phi}_{I-1}, e_2) \in \mathcal{B}_\lambda$, *then there exists a constant* C *independent of* ε *such that the following estimates hold:*

$$\begin{aligned}&\big\|\mathcal{N}_\varepsilon(\Psi_1^\flat, \Psi_1^*, \Phi_{I-1}, e_1) - \mathcal{N}_\varepsilon(\Psi_2^\flat, \Psi_2^*, \widetilde{\Phi}_{I-1}, e_2)\big\|_{C_0^{2,\alpha}(M_\varepsilon)}\\ &\leq C\varepsilon^{I+1}\Big(\|\Psi_1^\flat - \Psi_2^\flat\|_{2,\varepsilon,\alpha} + \|\Psi_1^* - \Psi_2^*\|_{2,\varepsilon,\alpha,\rho} + \|\Phi_{I-1} - \widetilde{\Phi}_{I-1}\|_{2,\alpha} + \|e_1 - e_2\|_*\Big),\end{aligned}$$

$$\begin{aligned}&\big\|\Pi^\perp[\mathcal{M}_\varepsilon(\Psi_1^\flat, \Psi_1^*, \Phi_{I-1}, e_1)] - \Pi^\perp[\mathcal{M}_\varepsilon(\Psi_2^\flat, \Psi_2^*, \widetilde{\Phi}_{I-1}, e_2)]\big\|_{\varepsilon,\alpha,\rho}\\ &\leq C\varepsilon^{I+1}\Big(\|\Psi_1^\flat - \Psi_2^\flat\|_{2,\varepsilon,\alpha} + \|\Psi_1^* - \Psi_2^*\|_{2,\varepsilon,\alpha,\rho} + \|\Phi_{I-1} - \widetilde{\Phi}_{I-1}\|_{2,\alpha} + \|e_1 - e_2\|_*\Big),\end{aligned}$$

$$\begin{aligned}&\big\|\widetilde{\mathcal{M}}_{\varepsilon,1}(\Psi_1^\flat, \Psi_1^*, \Phi_{I-1}, e_1) - \widetilde{\mathcal{M}}_{\varepsilon,1}(\Psi_2^\flat, \Psi_2^*, \widetilde{\Phi}_{I-1}, e_2)\big\|_{C^{0,\alpha}(K)}\\ &\leq C\varepsilon^{I+2}\Big(\|\Psi_1^\flat - \Psi_2^\flat\|_{2,\varepsilon,\alpha} + \|\Psi_1^* - \Psi_2^*\|_{2,\varepsilon,\alpha,\rho} + \|\Phi_{I-1} - \widetilde{\Phi}_{I-1}\|_{2,\alpha} + \|e_1 - e_2\|_*\Big),\end{aligned}$$

and

$$\begin{aligned}&\big\|\widetilde{\mathcal{M}}_{\varepsilon,2}(\Psi_1^\flat, \Psi_1^*, \Phi_{I-1}, e_1) - \widetilde{\mathcal{M}}_{\varepsilon,2}(\Psi_2^\flat, \Psi_2^*, \widetilde{\Phi}_{I-1}, e_2)\big\|_{C^{0,\alpha}(K)}\\ &\leq C\varepsilon^{I+1}\Big(\|\Psi_1^\flat - \Psi_2^\flat\|_{2,\varepsilon,\alpha} + \|\Psi_1^* - \Psi_2^*\|_{2,\varepsilon,\alpha,\rho} + \|\Phi_{I-1} - \widetilde{\Phi}_{I-1}\|_{2,\alpha} + \|e_1 - e_2\|_*\Big).\end{aligned}$$

Proof. This proof is rather technical but does not offer any real difficulty. We mention here that the choice of the norm $\|\Psi^\flat\|_{2,\varepsilon,\alpha}$ is crucial to estimate the term

$$-\eta_\delta^\varepsilon\, h^{-p} p|W-1|^{p-2}(W-1)\Psi^\flat \qquad \text{in } \mathcal{M}_\varepsilon(\Psi^\flat, \Psi^*, \Phi_{I-1}, e).$$

We omit details here referring to [13, 29] and some references therein. □

Using the estimates in Lemmas 4.5 and 4.6, it is not difficult to prove that taking $I \geq 3k+1$ and λ sufficiently large in (4.40), then system (4.37) possesses a fixed point in $\mathcal{B}_\lambda$. Theorem 1.1 then follows at once. □

5. Appendices

5.1. **Appendix A: Proof of Proposition 2.1.** The proof is based on the Taylor expansion of the metric coefficients. We recall that the Laplace-Beltrami operator is given by

$$\Delta_g u = \frac{1}{\sqrt{\det g}}\,\partial_\alpha\Big(\sqrt{\det g}\, g^{\alpha\beta}\,\partial_\beta u\Big)$$

which can be rewritten as

$$\Delta_g u = g^{\alpha\beta}\partial^2_{\alpha\beta}u + (\partial_\alpha g^{\alpha\beta})\partial_\beta u + \frac{1}{2}\, g^{\alpha\beta}\partial_\alpha(\log\det g)\partial_\beta u.$$

Using the expansion of the metric coefficients determined above, we can easily prove that

$$\begin{aligned}g^{\alpha\beta}\,\partial^2_{\alpha\beta}u &= \tilde{g}^{ab}\,\partial^2_{ab}u + \partial^2_{ii}u + \varepsilon\left\{\tilde{g}^{cb}\,\Gamma^a_{ci} + \tilde{g}^{ca}\,\Gamma^b_{ci}\right\}(\xi^i + \Phi^i)\,\tilde{g}^{ab}\,\partial^2_{ab}u - 2\varepsilon\,\tilde{g}^{ab}\,\partial_{\bar b}\Phi^j\,\partial^2_{aj}u\\ &+ \varepsilon^2\Big(\tilde{g}^{ac}\,\Gamma^b_{dk}\Gamma^d_{cl} + \tilde{g}^{bc}\,\Gamma^a_{dk}\Gamma^d_{cl} + \tilde{g}^{cd}\,\Gamma^a_{dk}\Gamma^b_{cl}\Big)(\xi^k + \Phi^k)(\xi^l + \Phi^l)\,\partial^2_{ab}u\\ &+ 2\varepsilon^2\partial_{\bar b}\Phi^j\left\{\tilde{g}^{bc}\,\Gamma^a_{ci} + \tilde{g}^{ac}\,\Gamma^b_{ci}\right\}(\xi^i + \Phi^i)\,\partial^2_{aj}u + \varepsilon^2\,\tilde{g}^{ab}\,\partial_{\bar a}\Phi^i\partial_{\bar b}\Phi^j\,\partial^2_{ij}u\\ &+ R_3(\xi, \Phi, \nabla\Phi)(\partial^2_{ij}u + \partial^2_{aj}u + \partial^2_{ab}u).\end{aligned}$$

An easy computations yields

$$\partial_b g^{ab} = \partial_b \widetilde{g}^{ab} + \varepsilon^2 \, \partial_{\bar b} \Big\{ \widetilde{g}^{cb} \, \Gamma^a_{ci} + \widetilde{g}^{ca} \, \Gamma^b_{ci} \Big\} \, (\xi^i + \Phi^i) + \varepsilon^2 \, \Big\{ \widetilde{g}^{cb} \, \Gamma^a_{ci} + \widetilde{g}^{ca} \, \Gamma^b_{ci} \Big\} \, \partial_{\bar b} \, \Phi^i$$
$$+ R_3(\xi, \Phi, \nabla \Phi, \nabla^2 \Phi),$$

$$\partial_j g^{ja} = \varepsilon^2 \partial_{\bar b} \Phi^j \, \Big\{ \widetilde{g}^{bc} \, \Gamma^a_{cj} + \widetilde{g}^{ac} \, \Gamma^b_{cj} \Big\} + R_3(\xi, \Phi, \nabla \Phi),$$

$$\partial_a g^{aj} = -\varepsilon^2 \, \partial_{\bar a} \widetilde{g}^{ab} \, \partial_{\bar b} \Phi^j - \varepsilon^2 \, \widetilde{g}^{ab} \, \partial^2_{\bar a \bar b} \Phi^j + \varepsilon^3 \partial^2_{\bar a \bar b} \Phi^j \, \Big\{ \widetilde{g}^{bc} \, \Gamma^a_{ci} + \widetilde{g}^{ac} \, \Gamma^b_{ci} \Big\} \, (\xi^i + \Phi^i)$$
$$+ R_3(\xi, \Phi, \nabla \Phi, \nabla^2 \Phi),$$

$$\partial_i g^{ij} = R_3(\xi, \Phi, \nabla \Phi).$$

Then the following expansion holds

$$\begin{aligned}
&(\partial_\alpha g^{\alpha\beta}) \partial_\beta u = \\
&\partial_b \widetilde{g}^{ab} \, \partial_a u + \varepsilon^2 \, \partial_{\bar b} \Big\{ \widetilde{g}^{cb} \, \Gamma^a_{ci} + \widetilde{g}^{ca} \, \Gamma^b_{ci} \Big\} \, (\xi^i + \Phi^i) \, \partial_a u + \varepsilon^2 \, \Big\{ \widetilde{g}^{cb} \, \Gamma^a_{ci} + \widetilde{g}^{ca} \, \Gamma^b_{ci} \Big\} \, \partial_{\bar b} \, \Phi^i \, \partial_a u \\
&+ \varepsilon^2 \partial_{\bar b} \Phi^j \, \Big\{ \widetilde{g}^{bc} \, \Gamma^a_{cj} + \widetilde{g}^{ac} \, \Gamma^b_{cj} \Big\} \, \partial_a u - \varepsilon^2 \, \partial_{\bar a} \widetilde{g}^{ab} \, \partial_{\bar b} \Phi^j \, \partial_j u - \varepsilon^2 \, \widetilde{g}^{ab} \, \partial^2_{\bar a \bar b} \Phi^j \, \partial_j u \\
&+ \varepsilon^3 \partial^2_{\bar a \bar b} \Phi^j \, \Big\{ \widetilde{g}^{bc} \, \Gamma^a_{ci} + \widetilde{g}^{ac} \, \Gamma^b_{ci} \Big\} \, (\xi^i + \Phi^i) \, \partial_j u + R_3(\xi, \Phi, \nabla \Phi, \nabla^2 \Phi)(\partial_j u + \partial_a u).
\end{aligned}$$

On the other hand using the expansion of the log of determinant of g given in Lemma **??**, we obtain

$$\partial_b \log \big(\det g \big) = \partial_b \log \big(\det \widetilde{g} \big) - 2\varepsilon^2 \, \partial_{\bar b} (\Gamma^a_{ak}) \, (\xi^k + \Phi^k) - 2\varepsilon^2 \, \Gamma^a_{ak} \, \partial_{\bar b} \Phi^k + R_3(\xi, \Phi, \nabla \Phi, \nabla^2 \Phi).$$

and

$$\partial_i (\log \det g) = -2\varepsilon \, \Gamma^b_{bi} - 2\varepsilon^2 \, \Gamma^c_{ak} \Gamma^a_{ci} \, (\xi^k + \Phi^k) + R_3(\xi, \Phi, \nabla \Phi),$$

which implies that

$$\begin{aligned}
& \frac{1}{2} g^{\alpha\beta} \partial_\alpha (\log \det g) \partial_\beta u \\
= \;& \frac{1}{2} \, \partial_a (\log \det \widetilde{g}) \, \Big(\widetilde{g}^{ab} \, \partial_b u + \varepsilon \{ \widetilde{g}^{cb} \Gamma^a_{ci} + \widetilde{g}^{ca} \Gamma^b_{ci} \} (\xi^i + \Phi^i) \partial_b u - \varepsilon \, \widetilde{g}^{ab} \partial_{\bar b} \Phi^j \partial_j u \Big) \\
- \;& \varepsilon \, \Gamma^b_{bi} \partial_i u - \varepsilon^2 \, \Gamma^c_{ak} \Gamma^a_{ci} \, (\xi^k + \Phi^k) \, \partial_i u - \varepsilon^2 \, \Big(\partial_{\bar b} (\Gamma^d_{dk}) \, (\xi^k + \Phi^k) + \Gamma^d_{dk} \, \partial_{\bar b} \Phi^k \Big) \, \widetilde{g}^{ab} \partial_a u \\
+ \;& R_3(\xi, \Phi, \nabla \Phi, \nabla^2 \Phi)(\partial_j u + \partial_a u).
\end{aligned}$$

Collecting the above terms and recalling that

$$\Delta_{K_\varepsilon} u = \widetilde{g}^{ab} \partial^2_{ab} u + (\partial_a \widetilde{g}^{ab}) \partial_b u + \frac{1}{2} \, \widetilde{g}^{ab} \partial_a (\log \det \widetilde{g}) \partial_b u,$$

the desired result then follows at once.

5.2. **Appendix B: Proof of** (3.21)**.** We prove in this section the following identities
$$\int_{\mathbb{R}^N}(|1-U|^{p-2}(1-U)-1)\xi^i\partial_i U d\bar{\xi} = N\int_{\mathbb{R}^N} U d\bar{\xi} \quad \text{and} \quad \int_{\mathbb{R}^N} U d\bar{\xi} = \frac{\sigma}{N}\int_{\mathbb{R}^N}|\partial_1 U|^2 d\bar{\xi}$$
where $\sigma = \dfrac{p-1}{p}(\frac{p+1}{p-1}-\frac{n-k}{2})$.
We use Pohozaev identity, we have that
$$(2-N)\int Uf(U)dx = -2N\int F(U)dx$$
where
$$F(t) := \int_0^t f(s)ds \qquad \text{with} \qquad f(u) := |u-1|^p - 1.$$
We have $F(U) = \dfrac{1}{p+1}\Big(|U-1|^p(U-1)+1\Big) - U$. Then
$$(2-N)\int U(|U-1|^p-1)dx = -2N\int \frac{1}{p+1}\Big(|U-1|^p(U-1)+1\Big) - U dx.$$
Namely
$$\begin{aligned}(2-N)\int|U-1|^pU\,dx-(2-N)\int U\,dx &= 2N\int Udx - \frac{2N}{p+1}\int|U-1|^pU\,dx\\ &+ \frac{2N}{p+1}\int(|U-1|^p-1)\,dx.\end{aligned}$$
Using the fact that
$$\int(|U-1|^p-1) = \int \Delta U = 0$$
we get
$$\int U\,dx = \frac{(p+1)(2-N)+2N}{(p+1)(N+2)}\int|U-1|^pU\,dx. \tag{5.1}$$
On the other hand, since U satisfies
$$-\Delta U = |U-1|^p - 1,$$
using integration by parts (multiply by U), we have that
$$\int|\nabla U|^2\,dx = \int|U-1|^pU - \int U\,dx$$
Remplacing this equality in (5.1), we obtain
$$\int U\,dx = \frac{\sigma}{N}\int|\nabla U|^2\,dx.$$
Concerning the second equality, we have
$$\begin{aligned}&\int_{\mathbb{R}^N}(|1-U|^{p-2}(1-U)-1) < \bar{\xi},\nabla U > d\bar{\xi}\\ =\ &\int_{\mathbb{R}^N}(|1-U|^{p-2}(1-U)) < \bar{\xi},\nabla U > d\bar{\xi} - \int_{\mathbb{R}^N} < \bar{\xi},\nabla U > d\bar{\xi}\\ =\ &\int_{\mathbb{R}^N} < \bar{\xi},(|1-U|^{p-2}(1-U))\nabla U > d\bar{\xi} + N\int_{\mathbb{R}^N} U\,d\bar{\xi}\\ =\ &\int_{\mathbb{R}^N} < \bar{\xi},\nabla(|1-U|^p) > d\bar{\xi} + N\int_{\mathbb{R}^N} U\,d\bar{\xi}\\ =\ &\int_{\mathbb{R}^N} < \bar{\xi},\nabla(|1-U|^p-1) > d\bar{\xi} + N\int_{\mathbb{R}^N} U\,d\bar{\xi}\\ =\ &-\frac{N}{p}\int_{\mathbb{R}^N}(|1-U|^p-1)d\bar{\xi} + N\int_{\mathbb{R}^N} U\,d\bar{\xi}\end{aligned}$$

Using again the fact that

$$\int_{\mathbb{R}^N} (|1-U|^p - 1)d\bar{\xi} = \int_{\mathbb{R}^N} \Delta U d\bar{\xi} = 0$$

it follows that

$$\int_{\mathbb{R}^N} (|1-U|^{p-2}(1-U) - 1) < \bar{\xi}, \nabla U > d\bar{\xi} = N \int_{\mathbb{R}^N} U \, d\bar{\xi}.$$

This ends the proof of (3.21).

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