Simultaneous Blow-up for two species Patlack-Keller-Segel System in $\mathbb{R}^2$.

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Abstract

The collective synchronised movement of unicellular organisms attracting or repelling by chemical signals are defined as chemotaxis. Evelyn Fox Keller and Lee Segel in [4] proposed one Parabolic-Parabolic system (PKS) to model chemotaxis. The general PKS system is the parabolic-parabolic system

$$
\begin{align*}
 c_t &= \nabla \cdot (k_1(c,v)\nabla c - k_2(c,v)c\nabla v) + k_3(c,v), \\
v_t &= D_\nu \Delta v + k_4(c,v) - k_5(c,v)v,
\end{align*}
$$

(1)

where $x \in \Omega \subset \mathbb{R}^n$, $t \in \mathbb{R}^+_0$ and the biological meaning of variables are: $c$ organism density, $v$ chemical concentration, $k_1$ organism kinetic movement, $k_2$ chemical sensibility, $k_3$ organism fitness function, $D_\nu$ chemical diffusion, $k_4$ chemical production, $k_5$ chemical degradation, subject to Neumann or Dirichlet boundary conditions and smooth boundary $\partial \Omega$, positive initial data are considered $u(x,0) = u_0$ and $v(x,0) = v_0$. It was conjectured by S. Childress & J.K. Percus in [1], that in a two-dimensional domain there exists a critical number $C$ such that if $\int u_0(x)dx < C$ then the solution exists globally in time and if $\int u_0(x)dx > C$ blow-up happens. To different versions of the Keller-Segel model the conjecture has been essentially proved, finding the critical value $C = 8\pi/\chi$.

In the case of several chemotactic species a new question arise with the objective to extend the S. Childress & J.K. Percus’s conjecture,

**Is there a critical curve in the plane of initial masses $\theta_1\theta_2$ delimiting on one side global existence and blow-up on the other side?**

Let consider the general PKS parabolic-elliptic system,

$$
\begin{align*}
 \partial_t u_1 &= \Delta u_1 - \chi_1 \nabla (u_1 \nabla v_1) - \chi_2 \nabla (u_1 \nabla v_2) \\
 \partial_t u_2 &= \Delta u_2 - \chi_3 \nabla (u_2 \nabla v_1) - \chi_4 \nabla (u_2 \nabla v_2) \\
 \quad - \Delta v_1 &= \alpha_1 u_1 + \alpha_2 u_2 \\
 \quad - \Delta v_2 &= \alpha_3 u_1 + \alpha_4 u_2
\end{align*}
$$

(2)

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where \( x \in \mathbb{R}^2, t > 0, u_1(x, 0), u_2(x, 0) \in L^1(\mathbb{R}^2, (1 + |x|^2)dx) \). The biological meaning of variables are: \( u_i \) is the population density, \( v_i \) represent the chemo-attractant concentration, \( \chi_i \) population sensibility at the chemical substance. The first optimal response to this question for one simplified version of (2) it was given in [2,3] and consider two positive chemotactic populations producing the same chemo-attractant and each population have one different sensibility to the chemical signal. The sharp curves funded in [2,3] are defined by

\[
8\pi \left( \frac{\theta_1}{\chi_1} \mu + \frac{\theta_2}{\chi_2} \right) - (\theta_1 + \theta_2)^2 = 0, \quad \theta_1 = \frac{8\pi}{\chi_1}, \quad \theta_2 = \frac{8\pi}{\chi_2}\]

Arising a new open question proposed in [2,3]

Is the Blow-up for \( u_i \) and \( u_2 \) be simultaneous?

The simultaneous Blow-up occurs when \( u_1 \) and \( u_2 \) have the same existence time. The method applied to prove the simultaneous Blow-up consist in the following step:

1. The entropy comparison criterion which implies that if one is bounded then another entropy too.
2. The Vallée Poussin criterion to prove uniform integrable or equi-integrable of solutions.
3. Equi-integrability porperty permits to obtain \( L^p \) bounds for all \( p \in (1, +\infty) \).
4. The Moser-Trudinger and Zigmud-Calderón inequalities to obtain \( L^\infty \) bound.

The main result is: **if one solution of (2) have bounded entropy, then the two solutions are bounded in \( L^\infty(0, T) \) or equivalently the Blow-up should be simultaneous.**

References


