Simultaneous Blow-up for two species Patlack-Keller-Segel System in $\mathbb{R}^2$.

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Abstract

The collective synchronised movement of unicellular organisms attracting or repelling by chemical signals are defined as chemotaxis. Evelyn Fox Keller and Lee Segel in [4] proposed one Parabolic-Parabolic system (PKS) to model chemotaxis. The general PKS system is the parabolic-parabolic system

$$\left.\begin{array}{rcl} c_t & = & \nabla\cdot(k_1(c,v)\nabla c - k_2(c,v)c\nabla v) + k_3(c,v), \\ \epsilon v_t & = & D_v\Delta v + k_4(c,v) - k_5(c,v)v, \end{array}\right\} \tag{1}$$

where $x \in \Omega \subset \mathbb{R}^n$, $t \in \mathbb{R}_0^+$ and the biological meaning of variables are: c organism density, v chemical concentration, k_1 organism kinetic movement, k_2 chemical sensibility, k_3 organism fitness function, D_v chemical diffusion, k_4 chemical production, k_5 chemical degradation, subject to Neumann or Diritchlet boundary conditions and smooth boundary $\partial\Omega$, positive initial data are considered $u(x,0) = u_0$ and $v(x,0) = v_0$. It was conjectured by S. Childress & J.K. Percus in [1], that in atwo-dimensional domain there exists a critical number C such that if $\int u_0(x)dx < C$ then the solution exists globally in time and if $\int u_0(x)dx > C$ blow-up happens. To different versions of the Keller-Segel model the conjecture has been essentially proved, finding the critical value $C = 8\pi/\chi$.
In the case of several chemotactic species a new question arise with the objective to extend the S. Childress & J.K. Percus's conjeture,

Is there a critical curve in the plane of initial masses $\theta_1\theta_2$ delimiting on one side global existence and blow-up on the other side?

Let consider the general PKS parabolic-elliptic system,

$$\left.\begin{array}{r} \partial_t u_1 = \triangle u_1 - \chi_1\nabla(u_1\nabla v_1) - \chi_2\nabla(u_1\nabla v_2) \\ \partial_t u_2 = \triangle u_2 - \chi_3\nabla(u_2\nabla v_1) - \chi_4\nabla(u_2\nabla v_2) \\ -\triangle v_1 = \alpha_1 u_1 + \alpha_2 u_2 \\ -\triangle v_2 = \alpha_3 u_1 + \alpha_4 u_2 \end{array}\right\} \tag{2}$$

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where $x \in \mathbb{R}^2$, $t > 0$, $u_1(x,0)$, $u_2(x,0) \in L^1(\mathbb{R}^2, (1+|x|^2)dx)$. The biological meaning of variables are: u_i is the population density, v_i represent the chemo-attractant concentration, χ_i population sensibility at the chemical substance.The first optimal response to this question for one simplified version of (2) it was given in [2, 3] and consider two positive chemotactic populations producing the same chemo-attractant and each population have one different sensibility to the chemical signal. The sharp curves funded in [2, 3] are defined by

$$8\pi\left(\frac{\theta_1}{\chi_1}\mu + \frac{\theta_2}{\chi_2}\right) - (\theta_1+\theta_2)^2 = 0, \quad \theta_1 = \tfrac{8\pi}{\chi_{1,}} \quad \theta_2 = \frac{8\pi}{\chi_2}. \tag{3}$$

Arising a new open question proposed in [2, 3]

Is the Blow-up for u_i and u_2 be simultaneous?

The simultaneous Blow-up occurs when u_1 and u_2 have the same existence time. The method applied to prove the simultaneous Blow-up consist in the following step:

1. The entropy comparison criterion which implies that if one is bounded then another entropy too.
2. The Vallée Poussin criterion to prove uniform integrable or equi-integrable of solutions.
3. Equi-integrability porperty permits to obtain L^p bounds for all $p \in (1, +\infty)$.
4. The Moser-Trudinger and Zigmud-Calderón inequalities to obtain L^∞ bound.
5. The Alikatos-Moser iteration presented in [5] over non bounded Domain.

The main result is: **if one solution of** (2) **have bounded entropy, then the two solutions are bounded in** $L^\infty(0,T)$ or equivalently the Blow-up should be simultaneous.

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