BUBBLE TOWER SOLUTIONS FOR SUPERCRITICAL ELLIPTIC PROBLEM IN $\mathbb{R}^N$

WENJING CHEN, JUAN DÁVILA, AND IGNACIO GUERRA

Abstract. We consider the following problem

$$\begin{cases} -\Delta u + u = u^p + \lambda u^q, \quad u > 0 \quad \text{in } \mathbb{R}^N; \\ u(z) \to 0 \quad \text{as} \quad |z| \to \infty, \end{cases}$$

where $p = p^* + \varepsilon$, with $p^* = \frac{N+2}{N-2}$, $1 < q < \frac{N+2}{N-2}$ if $N \geq 4$, $3 < q < 5$ if $N = 3$, $\lambda > 0$, and ε is a positive parameter. We prove that for $\varepsilon > 0$ small enough, it has a solution with the shape of a tower of bubbles.

Keywords: elliptic equation, non-uniqueness, bubble-tower solutions

1. Introduction

We are interested in the elliptic equation

$$\begin{cases} -\Delta u + u = u^p + \lambda u^q, \qquad u > 0 \qquad \text{in } \mathbb{R}^N, \\ u(x) \to 0 \quad \text{as } |x| \to \infty, \end{cases} \tag{1.1}$$

where $N \geq 3$, $\lambda > 0$ and $1 < q < p$. This problem arises in the study of standing waves of a nonlinear Schrödinger equation with two power type nonlinearities, see for example Tao, Visan and Zhang [28].

If $p = q$, equation (1.1) reduces to

$$\begin{cases} -\Delta u + u = u^p, \qquad u > 0 \qquad \text{in } \mathbb{R}^N, \\ u(x) \to 0 \quad \text{as } |x| \to \infty, \end{cases} \tag{1.2}$$

after a suitable scaling.

Thanks to the classical result of Gidas, Ni and Nirenberg [15], solutions of (1.1) and (1.2) are radially symmetric about some point, which we will assume is always the origin.

It is well known that problem (1.2) has a solution if and only if $1 < p < \frac{N+2}{N-2}$. Existence was proved by Berestycki and Lions [2], while non-existence follows from the Pohozaev identity [26]. Uniqueness also holds and was fully settled by Kwong [16], after a series of contributions [4, 17, 23, 24, 22, 21]. See also Felmer, Quaas, Tang and Yu [10] for further properties.

Concerning (1.1), the work of Berestycki and Lions [2] is still applicable if $1 < q < p < \frac{N+2}{N-2}$, and one obtains existence of a solution. If $p, q \geq \frac{N+2}{N-2}$ there is no solution, again from the Pohozaev identity.

Recently, Dávila, del Pino and Guerra [5] proved that uniqueness does not hold in general for (1.1), if $1 < q < p < \frac{N+2}{N-2}$. More precisely if $N = 3$, the authors obtained at least three solutions to problem (1.1) if $1 < q < 3$, $\lambda > 0$ is sufficiently large and fixed, and $p < 5$ is close enough to 5.

Let us mention some contributions to the question of existence for (1.1) when one exponent is subcritical and other is critical or supercritical. If $1 < q < p = \frac{N+2}{N-2}$ in (1.1), Alves, de Morais Filho and Souto [1] proved:

- when $N \geq 4$, there exists a nontrivial classical solution for all $\lambda > 0$ and $1 < q < \frac{N+2}{N-2}$;
- when $N = 3$, there exists a nontrivial classical solution for all $\lambda > 0$ and $3 < q < 5$;
- when $N = 3$, there exists a nontrivial classical solution for $\lambda > 0$ large enough and $1 < q \leq 3$.

Moreover, Ferrero and Gazzola [11] proved that for $q < \frac{N+2}{N-2} \leq p$, there exists $\bar{\lambda} > 0$, such that if $\lambda > \bar{\lambda}$, then (1.1) has at least one solution, while for $q < \frac{N+2}{N-2} < p$, there exists $0 < \underline{\lambda} < \bar{\lambda}$ such that if $\lambda < \underline{\lambda}$, then there is no solution.

In this paper, we are interested in multiplicity of solutions of (1.1), and for this we take an asymptotic approach, that is, we consider

$$\begin{cases} -\Delta u + u = u^p + \lambda u^q, \quad u > 0 \qquad \text{in } \mathbb{R}^N, \\ u(z) \to 0 \quad \text{as} \quad |z| \to \infty, \end{cases} \tag{1.3}$$

where $p = p^* + \varepsilon$, with $p^* = \frac{N+2}{N-2}$, $\lambda > 0$ and $\varepsilon > 0$ are parameters, and q satisfies

$$1 < q < \frac{N+2}{N-2} \quad \text{if } N \geq 4; \qquad 3 < q < 5 \quad \text{if } N = 3. \tag{1.4}$$

Our result can be stated as follows.

Theorem 1.1. *Let $\lambda > 0$ and let q satisfy (1.4). Given an integer $k \geq 1$, then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, there is a solution $u_\varepsilon(z)$ of problem (1.3) of the form*

$$u_\varepsilon(z) = (N(N-2))^{\frac{N-2}{4}} \sum_{j=1}^{k} \frac{\varepsilon^{-[(j-1)+\frac{1}{p^*-q}]}(\Lambda_j^*)^{-\frac{N-2}{2}}}{\left(1 + \varepsilon^{-\frac{4}{N-2}[(j-1)+\frac{1}{p^*-q}]}(\Lambda_j^*)^{-2}|z|^2\right)^{\frac{N-2}{2}}}(1 + o(1)), \tag{1.5}$$

where the constants $\Lambda_j^* > 0$, $j = 1, 2, \ldots, k$, can be computed explicitly and depend on k, N, q.

The expansion (1.5) is valid if $\frac{1}{C}\varepsilon^{\frac{2}{N-2}[(i-1)+\frac{1}{p^*-q}]} \leq |z| \leq C\varepsilon^{\frac{2}{N-2}[(i-1)+\frac{1}{p^*-q}]}$, with some $i \in \{1, 2, \cdots, k\}$, and $o(1) \to 0$ uniformly as $\varepsilon \to 0$ in this region.

The solutions described in this result behave like a superposition of "bubbles" of different blow-up orders centered at the origin, and hence have been called bubble-tower solutions. By bubbles we mean the functions

$$w_\mu(z) = \alpha_N \frac{\mu^{\frac{N-2}{2}}}{(\mu^2 + |z|^2)^{\frac{N-2}{2}}}, \quad \text{with} \quad \alpha_N = (N(N-2))^{\frac{N-2}{4}}, \tag{1.6}$$

where $\mu > 0$, which are the unique positive solutions of

$$-\Delta w = w^{p^*} \quad \text{in } \mathbb{R}^N,$$

(except translations).

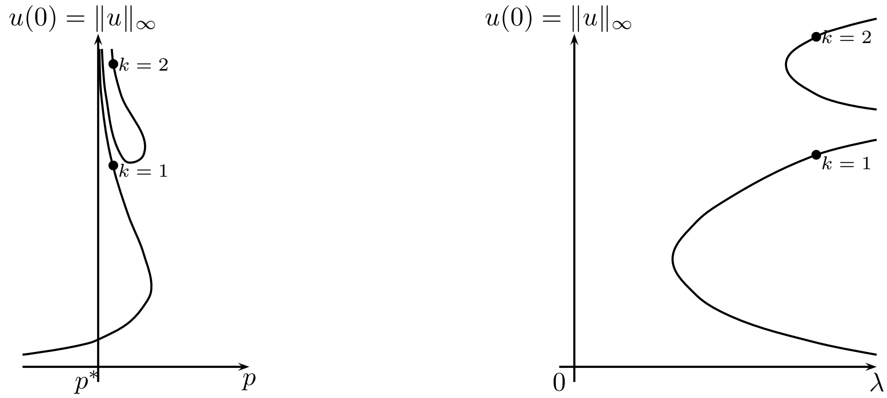

FIGURE 1. Left: $u(0)$ vs. p for λ large and fixed. Right: $u(0)$ vs. λ for $p = p^* + \varepsilon$, $\varepsilon > 0$ small and fixed.

Based on numerical simulations we present bifurcation diagrams for solutions of (1.3) where q satisfies (1.4). In Figure 1 (left) we show the bifurcation diagram as a function of p for a fixed large λ, and in Figure 1 (right) we show the diagram as a function of λ for $p = p^* + \varepsilon$, $\varepsilon > 0$ small and fixed. In both diagrams we observe branches of solutions, with the upper part having unbounded solutions as $\varepsilon \to 0$ or $\lambda \to \infty$. We believe that the solutions constructed in Theorem 1.1 are located on these upper branches, and are shown in the diagrams for the cases of 1 and 2 bubbles.

Bubble-tower solutions were found by del Pino, Dolbeault and Musso [6] for a slightly supercritical Brezis-Nirenberg problem in a ball, and after that have been studied intensively [3, 7, 8, 9, 13, 14, 18, 19, 20, 25]. In particular we mention the work of Campos [3] who considered the existence of bubble-tower solutions to a problem related to ours:

$$\begin{cases} -\Delta u = u^{p^* \pm \varepsilon} + u^q, \quad u > 0 \quad \text{in } \mathbb{R}^N; \\ u(z) \to 0 \ \text{ as } \ |z| \to \infty, \end{cases}$$

with $\frac{N}{N-2} < q < p^* = \frac{N+2}{N-2}$, $N \geq 3$.

For the proof we consider a variation of the so-called Emden-Fowler transformation:

$$v(x) = \left(\frac{p^* - 1}{2}\right)^{\frac{2}{p^*-1}} r^{\frac{2}{p^*-1}} u(r),$$

with

$$r = |z| = e^{-\frac{p^*-1}{2}x}, \quad x \in (-\infty, +\infty).$$

Then finding a radial solution $u(r)$ to (1.3) corresponds to solving the problem

$$\begin{cases} \mathcal{L}_0(v) = \alpha_\varepsilon e^{\varepsilon x} v^{p^*+\varepsilon} + \lambda \beta_N e^{-(p^*-q)x} v^q \quad \text{in } (-\infty, +\infty); \\ v(x) > 0 \quad \text{for} \quad x \in (-\infty, +\infty); \\ v(x) \to 0 \quad \text{as} \quad |x| \to \infty, \end{cases} \tag{1.7}$$

where

$$\mathcal{L}_0(v) = -v'' + v + \left(\frac{2}{N-2}\right)^2 e^{-\frac{4}{N-2}x} v, \tag{1.8}$$

is the transformed operator associated to $-\Delta + Id$, and α_ε, β_N are constants.

Under the Emden-Fowler transformation the bubbles w_μ take the form

$$W(x-\xi) = \left(\frac{4N}{N-2}\right)^{\frac{N-2}{4}} e^{-(x-\xi)} \left(1 + e^{-\frac{4}{N-2}(x-\xi)}\right)^{-\frac{N-2}{2}} \tag{1.9}$$

with $\mu = e^{-\frac{2}{N-2}\xi}$, and solve

$$\begin{cases} W'' - W + W^{p^*} = 0, & \text{in } (-\infty, +\infty); \\ W'(0) = 0; \\ W(x) > 0, \quad W(x) \to 0 \quad \text{as} \quad |x| \to \infty. \end{cases}$$

In Section 2, we build an approximate solution to (1.7) as a sum of suitable projections of the transformed bubbles W centered at $0 < \xi_1 < \ldots < \xi_k$ with $\xi_1 \to \infty$. After the study of the linearized problem at the approximate solution in Section 3, and solvability of a nonlinear projected problem in Section 4, we perform a Lyapunov-Schmidt reduction procedure as in [12, 18, 3]. Then the problem becomes to find a critical point of some functional depending on $0 < \xi_1 < \ldots < \xi_k$. This is done in Section 5 where Theorem 1.1 is proved.

From the technical point of view, one difficulty is due to the form of the linearized operator. As $r \to \infty$ dominates $-\Delta + I$ (or $\mathcal{L}_0$ as $x \to -\infty$ after the change of variables) while near the regions of concentration the important part of the linearization is $\Delta + p^* w_\mu^{p^*-1}$. This is taken into account in the norm we use for the solutions of linearized problem, and it is more naturally written for the functions after the Lane-Emden transformation. This is different from may previous works, but is already contained in [5].

2. The first approximate solution

In this section, we build the first approximate solution to (1.3). In order to do this, we introduce U_μ as the unique solution of the following problem

$$\begin{cases} -\Delta U_\mu + U_\mu = w_\mu^{p^*} & \text{in } \mathbb{R}^N, \\ U_\mu(z) \to 0 \quad \text{as} \quad |z| \to \infty, \end{cases} \tag{2.1}$$

where w_μ are the bubbles (1.6). We write

$$U_\mu(z) = w_\mu(z) + R_\mu(z).$$

Then $R_\mu(z)$ satisfies

$$-\Delta R_\mu(z) + R_\mu(z) = -w_\mu(z) \quad \text{in } \mathbb{R}^N, \quad R_\mu(z) \to 0 \quad \text{as} \quad |z| \to \infty.$$

We have the following result, whose proof is postponed to the Appendix.

Lemma 2.1. *Assume $0 < \mu \le 1$, we have*

(a) $0 < U_\mu(z) \le w_\mu(z)$, *for* $z \in \mathbb{R}^N$.

(b) One has

$$U_\mu(z) \le C\mu^{\frac{N-2}{2}} |z|^{-(N+2)}, \qquad \text{for } |z| \ge R,$$

where R is a large positive number but fixed.

(c) Given any $\mu > 0$ small, we have

$$|R_\mu(z)| \leq C \frac{\mu^{\frac{N-2}{2}}}{|z|^{N-2}} \quad \text{for} \quad N \geq 3, \quad |z| \geq 1. \tag{2.2}$$

$$|R_\mu(z)| \quad \leq \quad C \begin{cases} \mu^{-\frac{N-6}{2}} & \text{for } N \geq 5; \\ \mu \log \frac{1}{\mu} & \text{for } N = 4; \\ \mu^{\frac{1}{2}} & \text{for } N = 3. \end{cases} \qquad |z| \leq \frac{\mu}{2}. \tag{2.3}$$

$$|R_\mu(z)| \quad \leq \quad C \begin{cases} \mu^{-\frac{N-6}{2}} \frac{1}{(1+|\frac{z}{\mu}|^2)^{\frac{N-4}{2}}} & \text{for } N \geq 5; \\ \mu \log \frac{1}{|z|} & \text{for } N = 4; \\ \mu^{\frac{1}{2}} & \text{for } N = 3. \end{cases} \qquad \frac{\mu}{2} \leq |z| \leq 1. \tag{2.4}$$

We define the following Emden-Fowler transformation

$$v(x) = \mathcal{T}(u(r)) = \left(\frac{p^*-1}{2}\right)^{\frac{2}{p^*-1}} r^{\frac{2}{p^*-1}} u(r), \quad r = |z| = e^{-\frac{p^*-1}{2}x}$$

with $x \in (-\infty, +\infty)$. Using this transformation, finding a radial solution $u(r)$ to problem (1.3) corresponds to that of solving the problem (1.7). where

$$\alpha_\varepsilon = \left(\frac{p^*-1}{2}\right)^{-\frac{2\varepsilon}{p^*-1}}, \qquad \beta_N = \left(\frac{p^*-1}{2}\right)^{\frac{2(p^*-q)}{p^*-1}}.$$

We observe that $\mathcal{L}_0$ is the transformed operator associated to $-\Delta + Id$.

Define

$$V_\xi(x) = \mathcal{T}(U_\mu)(r), \quad \text{with } r = e^{-\frac{p^*-1}{2}x}, \quad \mu = e^{-\frac{2}{N-2}\xi}.$$

Then $V_\xi(x)$ is the solution of the problem

$$\begin{cases} \mathcal{L}_0 V_\xi(x) = W(x-\xi)^{p^*} & \text{in } (-\infty, +\infty); \\ V_\xi(x) \to 0 \quad \text{as} \quad |x| \to \infty. \end{cases}$$

We write

$$V_\xi(x) = W(x-\xi) + R_\xi(x),$$

where W is given in (1.9) and $R_\xi(x) = \mathcal{T}(R_\mu)(r)$. By the Emden-Fowler transformation and as a consequence of Lemma 2.1, we have the following estimates.

Lemma 2.2. *For $\xi > 0$, we have*

(a) $0 < V_\xi(x) \leq W(x-\xi) = O(e^{-|x-\xi|})$, *for* $x \in \mathbb{R}$.

(b)

$$V_\xi(x) \leq C e^{\frac{N+6}{N-2}x} e^{-\xi}, \qquad \text{for} \quad -\infty < x \leq -\frac{N-2}{2} \log R, \tag{2.5}$$

for $R > 0$ is a fixed large number as Lemma 2.1.

(c) For $N \geq 3$, there is a positive constant C, such that

$$|R_\xi(x)| \leq C \begin{cases} e^{-|x-\xi|} & \text{if} \quad x \leq 0; \\ e^{-|x-\xi|} e^{-\frac{2}{N-2}\min\{x,\xi\}} & \text{if} \quad x \geq 0. \end{cases}$$

Define

$$Z_\xi(x) := \partial_\xi V_\xi(x) = \partial_\xi W(x-\xi) + \partial_\xi R_\xi(x).$$

Note that $\partial_\xi W(x-\xi) = O(e^{-|x-\xi|})$ and

$$\partial_\xi W(x-\xi) = -\frac{2}{N-2}\mu \mathcal{T}\left(\partial_\mu w_\mu(r)\right),$$

$$Z_\xi(x) = -\frac{2}{N-2}\mu \mathcal{T}\left(\widetilde{Z}_\mu(r)\right) \qquad \text{with} \quad \widetilde{Z}_\mu(z) = \partial_\mu U_\mu(z), \tag{2.6}$$

$$\partial_\xi R_\xi(x) = -\frac{2}{N-2}\mu \mathcal{T}\left(\partial_\mu R_\mu(r)\right). \tag{2.7}$$

Then from (6.1), (2.7) and Lemma 2.2 (c), we have for $N \geq 3$,

$$|\partial_\xi R_\xi(x)| \leq C \begin{cases} e^{-|x-\xi|} & \text{if} \quad x \leq 0; \\ e^{-|x-\xi|} e^{-\frac{2}{N-2}\min\{x,\xi\}} & \text{if} \quad x \geq 0. \end{cases}$$

Therefore

$$Z_\xi(x) = O(e^{-|x-\xi|}) \quad \text{for } \forall\, x \in \mathbb{R}.$$

Moreover, from (6.2) and (2.6), we find

$$|Z_\xi(x)| \leq C e^{\frac{N+6}{N-2}x} e^{-\xi}, \qquad \text{for} \quad -\infty < x \leq -\frac{N-2}{2}\log R,$$

for a fixed large $R > 0$.

Let $\eta > 0$ be a small but fixed number. Given an integer number k, let Λ_j, for $j = 1, \cdots, k$, be positive numbers and satisfy

$$\eta < \Lambda_j < \frac{1}{\eta}. \tag{2.8}$$

Set

$$\mu_1 = \varepsilon^{\frac{2}{(N+2)-(N-2)q}} \Lambda_1 \quad \text{and} \quad \mu_j = \varepsilon^{\frac{2}{N-2}(j-1) + \frac{2}{(N+2)-(N-2)q}} \Lambda_j \tag{2.9}$$

for $j = 2, \cdots, k$. We observe that

$$\frac{\mu_{j+1}}{\mu_j} = \varepsilon^{\frac{2}{N-2}} \frac{\Lambda_{j+1}}{\Lambda_j}, \qquad j = 1, \cdots, k-1.$$

Define k points in $\mathbb{R}$ as

$$\mu_j = e^{-\frac{2}{N-2}\xi_j}, \qquad j = 1, \cdots, k.$$

Then we have that

$$0 < \xi_1 < \xi_2 < \cdots < \xi_k,$$

and

$$\begin{cases} \xi_1 = -\frac{1}{p^*-q}\log\varepsilon - \frac{N-2}{2}\log\Lambda_1, \\ \xi_j - \xi_{j-1} = -\log\varepsilon - \frac{N-2}{2}\log\frac{\Lambda_j}{\Lambda_{j-1}}, \quad j = 2, \cdots, k. \end{cases} \tag{2.10}$$

Set

$$W_j = W(x-\xi_j), \quad R_j = R_{\xi_j}(x), \quad V_j = W_j + R_j, \quad V = \sum_{j=1}^{k} V_j. \tag{2.11}$$

We look for a solution of (1.3) of the form $u = \sum_{j=1}^{k} U_{\mu_j} + \psi$ corresponds to find a solution of (1.7) of the form $v = V + \phi$, where V is given by (2.11) and $\phi = \mathcal{T}(\psi)$ is a small term. Thus problem (1.7) becomes

$$\begin{cases} \mathcal{L}_\varepsilon(\phi) = N(\phi) + E & \text{in } (-\infty, +\infty); \\ \phi(x) > 0 \quad \text{for} \quad x \in (-\infty, +\infty); \\ \phi(x) \to 0 \quad \text{as} \quad |x| \to \infty, \end{cases} \tag{2.12}$$

where

$$\mathcal{L}_\varepsilon(\phi) = \mathcal{L}_0(\phi) - \alpha_\varepsilon (p^* + \varepsilon) e^{\varepsilon x} V^{p^*+\varepsilon-1}\phi - \lambda q \beta_N e^{-(p^*-q)x} V^{q-1}\phi,$$

$$\begin{aligned} N(\phi) &= \alpha_\varepsilon e^{\varepsilon x}\left[(V+\phi)^{p^*+\varepsilon} - V^{p^*+\varepsilon} - (p^*+\varepsilon)V^{p^*+\varepsilon-1}\phi\right] \\ &\quad + \lambda \beta_N e^{-(p^*-q)x}\left[(V+\phi)^q - V^q - qV^{q-1}\phi\right] \end{aligned}$$

and

$$\begin{aligned} E &= \alpha_\varepsilon e^{\varepsilon x} V^{p^*+\varepsilon} - \mathcal{L}_0(V) + \lambda \beta_N e^{-(p^*-q)x} V^q \\ &= \alpha_\varepsilon e^{\varepsilon x} V^{p^*+\varepsilon} - \sum_{j=1}^{k} W_j^{p^*} + \lambda \beta_N e^{-(p^*-q)x} V^q. \end{aligned}$$

where $\mathcal{L}_0$ is defined by (1.8).

3. The linear problem

In order to solve problem (2.12), we consider first the following problem: given points $\xi = (\xi_1, \cdots, \xi_k)$, finding a function ϕ such that for certain constants $c_1, c_2, \cdots, c_k$,

$$\begin{cases} \mathcal{L}_\varepsilon(\phi) = N(\phi) + E + \sum\limits_{j=1}^{k} c_j Z_j \quad \text{in } (-\infty, +\infty); \\ \lim\limits_{|x|\to\infty} \phi(x) = 0; \\ \int_{\mathbb{R}} Z_j \phi = 0, \quad \forall\, j = 1, \cdots, k, \end{cases} \tag{3.1}$$

where $Z_j(x) = Z_{\xi_j}(x) = \partial_{\xi_j} V_{\xi_j}(x)$ for $j = 1, 2, \cdots, k$.

To solve (3.1), it is important to understand its linear part, thus we consider the following problem: given a function h, finding ϕ such that

$$\begin{cases} \mathcal{L}_\varepsilon(\phi) = h + \sum\limits_{j=1}^{k} c_j Z_j \quad \text{in } (-\infty, +\infty); \\ \lim\limits_{|x|\to\infty} \phi(x) = 0; \\ \int_{\mathbb{R}} Z_j \phi = 0, \quad \forall\, j = 1, \cdots, k, \end{cases} \tag{3.2}$$

for certain constants c_j.

Now we analyze invertibility properties of the operator $\mathcal{L}_\varepsilon$ under the orthogonality conditions. Let σ satisfy

$$0 < \sigma < \min\left\{q - 1, 1, \frac{(N+2)(2q-1)}{N+6}, \frac{3q - p^*}{2}\right\}. \tag{3.3}$$

We define the real number M as follows

$$M = \begin{cases} 0, & \text{if } 1 \geq \frac{4}{N-2} + \sigma; \\ \max\{0, \gamma\}, & \text{if } 1 \leq \frac{4}{N-2} + \sigma, \end{cases} \tag{3.4}$$

where γ satisfies

$$\left(1-\left(\frac{4}{N-2}+\sigma\right)^2\right)e^{-\frac{4}{N-2}\gamma}=-\frac{1}{2}\left(\frac{2}{N-2}\right)^2.$$

We define the following norms for functions ϕ, h defined on $\mathbb{R}$,

$$\|\phi\|_*=\sup_{x\leq -M}e^{-(\frac{4}{N-2}+\sigma)x}e^{\sigma\xi_1}|\phi(x)|+\sup_{x\in\mathbb{R}}\left(\sum_{j=1}^k e^{-\sigma|x-\xi_j|}\right)^{-1}|\phi(x)|, \tag{3.5}$$

and

$$\|h\|_{**}\quad=\quad\sup_{x\in\mathbb{R}}\left(\sum_{j=1}^k e^{-\sigma|x-\xi_j|}\right)^{-1}|h(x)|.$$

The choice of norm here is motivated by the presence of 2 regimes in the solution of the linearized problem. Near the concentration points ξ_j we have a right hand side of the form $|h(x)|\leq Ce^{-\sigma|x-\xi_j|}$ and near these points the dominant terms in the linear operator $\mathcal{L}_\varepsilon$ are

$$-\phi''+\phi-\alpha_\varepsilon(p^*+\varepsilon)e^{\varepsilon x}V^{p^*+\varepsilon-1}\phi,$$

so we can expect the solution ϕ to be controlled by $|\phi(x)|\leq Ce^{-\sigma|x-\xi_j|}$. For $x\leq 0$ the dominant part of the linear operator is

$$\left(\frac{2}{N-2}\right)^2 e^{-\frac{4}{N-2}x}\phi.$$

Since the right hand side is controlled by $e^{-\sigma|x-\xi_1|}$, we can control ϕ using as supersolution $e^{(\frac{4}{N-2}+\sigma)x}e^{-\sigma\xi_1}$. Actually this will be a super solution for the whole linear operator for $x\leq -M$, where M is defined in (3.4).

The main result in this section is solvability of problem (3.2).

Proposition 3.1. *There exist positive numbers ε_0, and $C>0$ such that if the points $0<\xi_1<\xi_2<\cdots<\xi_k$ satisfy (2.10), then for all $0<\varepsilon<\varepsilon_0$ and all functions $h\in C(\mathbb{R};\mathbb{R})$ with $\|h\|_{**}<+\infty$, problem (3.2) has a unique solution $\phi=:T_\varepsilon(h)$ with $\|\phi\|_*<+\infty$. Moreover,*

$$\|\phi\|_*\leq C\|h\|_{**}\quad\text{and}\quad|c_j|\leq C\|h\|_{**}. \tag{3.6}$$

We first consider a simpler problem

$$\begin{cases}\mathcal{L}_0(\phi)-\alpha_\varepsilon(p^*+\varepsilon)e^{\varepsilon x}V^{p^*+\varepsilon-1}\phi=h+\sum\limits_{j=1}^k c_jZ_j \text{ in } (-\infty,+\infty);\\ \lim\limits_{|x|\to\infty}\phi(x)=0;\\ \int_{\mathbb{R}}Z_j\phi=0,\quad\forall\ j=1,\cdots,k,\end{cases} \tag{3.7}$$

for certain constants c_j, here $\mathcal{L}_0$ is defined by (1.8).

Lemma 3.2. *Under the assumptions of Proposition 3.1, then for all $0<\varepsilon<\varepsilon_0$ and any h, ϕ solution of (3.7), we have*

$$\|\phi\|_*\leq C\|h\|_{**}, \tag{3.8}$$

and

$$|c_j|\leq C\|h\|_{**}. \tag{3.9}$$

Proof. To prove (3.8), by contradiction, we suppose that there exist sequences ϕ_n, h_n, ε_n and c_j^n that satisfy (3.7), with

$$\|\phi_n\|_* = 1, \quad \|h_n\|_{**} \to 0, \quad \varepsilon_n \to 0.$$

We get a contradiction by the following steps.

Step 1: $c_j^n \to 0$ as $n \to +\infty$.

Multiplying (3.7) by Z_i^n and integrating by parts twice, we get that

$$\begin{aligned} &\sum_{j=1}^{k} c_j^n \int_{\mathbb{R}} Z_j^n Z_i^n \\ = &-\int_{\mathbb{R}} h_n Z_i^n + \int_{\mathbb{R}} \left[\mathcal{L}_0(Z_i^n) - \alpha_{\varepsilon_n}(p^* + \varepsilon_n) e^{\varepsilon_n x} V^{p^*+\varepsilon_n-1} Z_i^n\right] \phi_n. \end{aligned} \tag{3.10}$$

Note that

$$\int_{\mathbb{R}} Z_j^n Z_i^n = C\delta_{ij} + o(1),$$

where δ_{ij} is Kronecker's delta. Then (3.10) defines a linear system in the c_j's which is almost diagonal as $n \to \infty$.

Since $Z_i^n(x) = \partial_{\xi_i^n} V_{\xi_i^n}(x) = O(e^{-|x-\xi_i^n|})$, we then have

$$\begin{aligned} \left|\int_{\mathbb{R}} h_n Z_i^n\right| &\leq C\|h_n\|_{**} \int_{\mathbb{R}} \left(\sum_{j=1}^{k} e^{-\sigma|x-\xi_j^n|}\right) e^{-|x-\xi_i^n|} dx \\ &\leq Ck\|h_n\|_{**} \int_{\mathbb{R}} e^{-|y|} dy \leq C\|h_n\|_{**}. \end{aligned} \tag{3.11}$$

Moreover, Z_i^n satisfy

$$\mathcal{L}_0(Z_i^n) = p^* W^{p^*-1}(x - \xi_i^n) \partial_{\xi_i^n} W(x - \xi_i^n),$$

so we get

$$\left|\int_{\mathbb{R}} \left[\mathcal{L}_0(Z_i^n) - \alpha_{\varepsilon_n}(p^* + \varepsilon_n) e^{\varepsilon_n x} V^{p^*+\varepsilon_n-1} Z_i^n\right] \phi_n\right| = o(1)\|\phi_n\|_*. \tag{3.12}$$

From (3.10)-(3.12), we obtain

$$|c_j^n| \leq C\|h_n\|_{**} + o(1)\|\phi_n\|_*. \tag{3.13}$$

Thus $\lim_{n\to\infty} c_j^n = 0$.

Step 2: For any $L > 0$, any $l \in \{1, 2, \cdots, k\}$, we have

$$\sup_{x \in [\xi_l^n - L, \xi_l^n + L]} |\phi_n(x)| \to 0, \quad \text{as } n \to \infty. \tag{3.14}$$

Indeed, suppose not, we assume that there exist $L > 0$ and some $l \in \{1, 2, \cdots, k\}$ such that

$$|\phi_n(x_{n,l})| \geq c > 0, \quad \text{for some } x_{n,l} \in [\xi_l^n - L, \xi_l^n + L].$$

By elliptic estimates, there is a subsequence of ϕ_n converging uniformly on compact sets to a nontrivial bounded solution $\tilde{\phi}$ of

$$\mathcal{L}_0(\tilde{\phi}) = p^* W^{p^*-1}(x - \xi_l)\tilde{\phi},$$

where $\xi_l = \lim_{n\to\infty} \xi_l^n$. By nondegeneracy [27], it is well known that $\tilde{\phi} = cZ_l$ for some constant $c \neq 0$. But taking the limit in the orthogonality condition $\int_{\mathbb{R}} Z_l^n \phi_n = 0$, we obtain $\tilde{\phi} = 0$, which is a contradiction. Thus (3.14) holds.

Step 3: We prove that $\|\phi_n\|_* \to 0$ as $n \to \infty$.

Claim: For any $L > 0$ and $j \in \{1, 2, \cdots, k\}$, we have

$$\sup_{\mathbb{R}\setminus\cup_{j=1}^k[\xi_j^n - L, \xi_j^n + L]} \left(\sum_{j=1}^k e^{-\sigma|x-\xi_j^n|}\right)^{-1} |\phi_n(x)| \to 0, \tag{3.15}$$

and

$$\sup_{x\leq -M} e^{-(\frac{4}{N-2}+\sigma)x} e^{\sigma\xi_1^n} |\phi_n(x)| \to 0, \tag{3.16}$$

as $n \to +\infty$.

By the definition of $\|\cdot\|_*$ in (3.5), using (3.14), (3.15) and (3.16), we get that $\|\phi_n\|_* \to 0$ as $n \to \infty$.

Now we prove the above claim. We note that

$$h_n + \sum_{j=1}^k c_j^n Z_j^n \leq (C_0\|h_n\|_{**} + o(\|\phi_n\|_*)) \sum_{j=1}^k e^{-\sigma|x-\xi_j^n|}, \quad \text{with} \quad C_0 > 0.$$

For $x \in \mathbb{R}\setminus \cup_{j=1}^k [\xi_j^n - L, \xi_j^n + L]$, let us define

$$\begin{aligned} \tilde{\psi}_n(x) &= \left(C_0\|h_n\|_{**} + e^{\sigma L} \sup_{\cup_{j=1}^k[\xi_j^n-L,\xi_j^n+L]} |\phi_n(x)| + o(\|\phi_n\|_*)\right) \sum_{j=1}^k e^{-\sigma|x-\xi_j^n|} \\ &\quad + \varrho \sum_{j=1}^k e^{-\bar{\sigma}|x-\xi_j^n|} \end{aligned}$$

with $\varrho > 0$ small but fixed and $0 < \bar{\sigma} < \sigma$. Then by choosing suitable large $L > 0$, we get

$$\begin{aligned} & \mathcal{L}_0(\tilde{\psi}_n(x)) - \alpha_{\varepsilon_n}(p^* + \varepsilon_n) e^{\varepsilon_n x} V^{p^*+\varepsilon_n-1} \tilde{\psi}_n(x) \\ \geq\ & \mathcal{L}_0(\phi_n(x)) - \alpha_{\varepsilon_n}(p^* + \varepsilon_n) e^{\varepsilon_n x} V^{p^*+\varepsilon_n-1} \phi_n(x). \end{aligned}$$

On the other hand, we have that for any $L > 0$ and $j \in \{1, 2, \cdots, k\}$,

$$\tilde{\psi}_n(\xi_j^n - L) \geq \phi_n(\xi_j^n - L) \quad \text{and} \quad \tilde{\psi}_n(\xi_j^n + L) \geq \phi_n(\xi_j^n + L).$$

Moreover, there exists $R > 0$ large enough, such that

$$\tilde{\psi}_n(R) \geq \phi_n(R),$$

and

$$\tilde{\psi}_n(-R) \geq \phi_n(-R).$$

By the maximum principle, we get

$$\phi_n(x) \leq \tilde{\psi}_n(x) \quad \text{for} \quad x \in [-R, R]\setminus \cup_{j=1}^k [\xi_j^n - L, \xi_j^n + L].$$

Similarly, we obtain $\phi_n(x) \geq -\tilde{\psi}_n(x)$ for $x \in [-R, R]\setminus \cup_{j=1}^k [\xi_j^n - L, \xi_j^n + L]$. Thus

$$|\phi_n(x)| \leq \tilde{\psi}_n(x) \quad \text{for} \quad x \in [-R, R]\setminus \cup_{j=1}^k [\xi_j^n - L, \xi_j^n + L].$$

Letting $R \to +\infty$, we get

$$|\phi_n(x)| \leq \tilde{\psi}_n(x) \quad \text{for} \quad x \in \mathbb{R}\setminus \cup_{j=1}^k [\xi_j^n - L, \xi_j^n + L].$$

Letting $\varrho \to 0$, for $x \in \mathbb{R}\setminus \cup_{j=1}^{k} [\xi_j^n - L, \xi_j^n + L]$, we have that

$$|\phi_n(x)| \leq \left(C_0\|h_n\|_{**} + e^{\sigma L} \sup_{\cup_{j=1}^{k}[\xi_j^n - L, \xi_j^n + L]} |\phi_n(x)| + o(\|\phi_n\|_*) \right) \sum_{j=1}^{k} e^{-\sigma|x-\xi_j^n|}.$$

So (3.15) holds.

For $x \leq -M$, let $\rho > 0$ small and $C_1 > 0$ be chosen later on, we define

$$\psi_n(x) = C_1 \left(C_0\|h_n\|_{**} + o(\|\phi_n\|_*) \right) e^{(\frac{4}{N-2}+\sigma)x} e^{-\sigma\xi_1^n} + \rho e^{\frac{4}{N-2}x}.$$

By the maximum principle, we get

$$\phi_n(x) \leq \psi_n(x) \quad \text{for} \quad x \in [-R, -M],$$

if $R > 0$ is large enough. By a similar argument, we obtain $\phi_n(x) \geq -\psi_n(x)$ for $x \in [-R, -M]$. Thus

$$|\phi_n(x)| \leq \psi_n(x) \quad \text{for} \quad x \in [-R, -M].$$

Let $R \to +\infty$, we get

$$|\phi_n(x)| \leq \psi_n(x) \quad \text{for} \quad x \in [-\infty, -M].$$

Let $\rho \to 0$, we have

$$|\phi_n(x)| \leq C_1 \left(C_0\|h_n\|_{**} + o(\|\phi_n\|_*) \right) e^{(\frac{4}{N-2}+\sigma)x} e^{-\sigma\xi_1^n} \quad \text{for} \quad x \in [-\infty, -M].$$

So we obtain that (3.16) holds.

Moreover, estimate (3.9) follows from (3.13) and (3.8). □

Proof of Proposition 3.1. From Lemma 3.2, for ϕ and h satisfying (3.2), we then have

$$\|\phi\|_* \leq C \left(\|h\|_{**} + \|e^{-(p^*-q)x} V^{q-1}\phi\|_{**} \right),$$

and

$$|c_j| \leq C \left(\|h\|_{**} + \|e^{-(p^*-q)x} V^{q-1}\phi\|_{**} \right).$$

In order to establish (3.6), it is sufficient to show that

$$\|e^{-(p^*-q)x} V^{q-1}\phi\|_{**} \leq o(1)\|\phi\|_*. \tag{3.17}$$

Indeed,

$$\begin{aligned} \|e^{-(p^*-q)x} V^{q-1}\phi\|_{**} \quad &\leq \quad \sup_{x\leq -M} \left(\sum_{j=1}^{k} e^{-\sigma|x-\xi_j|} \right)^{-1} \left| e^{-(p^*-q)x} V^{q-1}\phi \right| \\ &\quad + \sup_{x\geq -M} \left(\sum_{j=1}^{k} e^{-\sigma|x-\xi_j|} \right)^{-1} \left| e^{-(p^*-q)x} V^{q-1}\phi \right| := Q_1 + Q_2. \end{aligned} \tag{3.18}$$

Now we estimate Q_1 and Q_2 respectively, we first have

$$\begin{aligned} Q_1 \quad &\leq \quad C \sup_{x\leq -M} e^{\sigma|x-\xi_1|} |\phi(x)| e^{-(p^*-q)x} V^{q-1} \\ &\leq \quad C e^{-(q-1)\xi_1} \sup_{x\leq -M} e^{-(\frac{4}{N-2}+\sigma)x} e^{\sigma\xi_1} |\phi(x)|. \end{aligned} \tag{3.19}$$

For Q_2, if $-M \le x \le \xi_1$, then we have

$$\begin{aligned} e^{-(p^*-q)x}V^{q-1} &\le \sum_{j=1}^{k} e^{-(p^*-q)x}e^{-(q-1)|x-\xi_j|} \le Ce^{(2q-p^*-1)x}e^{-(q-1)\xi_1} \\ &\le C\max\left\{e^{-(p^*-q)\xi_1}, e^{-(q-1)\xi_1}\right\}. \end{aligned}$$

If $x \ge \xi_1$, then we have

$$e^{-(p^*-q)x}V^{q-1} \le \sum_{j=1}^{k} e^{-(p^*-q)x}e^{-(q-1)|x-\xi_j|} \le Ce^{-(p^*-q)x} \le Ce^{-(p^*-q)\xi_1}.$$

Thus we find

$$Q_2 \le C\max\left\{e^{-(p^*-q)\xi_1}, e^{-(q-1)\xi_1}\right\} \sup_{x\ge -M}\left(\sum_{j=1}^{k} e^{-\sigma|x-\xi_j|}\right)^{-1}|\phi(x)|. \tag{3.20}$$

From (3.18), (3.19) and (3.20), we get

$$\|e^{-(p^*-q)x}V^{q-1}\phi\|_{**} \le C\max\left\{e^{-(p^*-q)\xi_1}, e^{-(q-1)\xi_1}\right\}\|\phi\|_* = o(1)\|\phi\|_*.$$

So estimate (3.17) holds.

We now prove the existence and uniqueness of solution to (3.2). Consider the Hilbert space

$$H = \left\{\phi \in H^1(\mathbb{R}) \ : \ \int_{\mathbb{R}} Z_j\phi = 0, \quad \forall\ j = 1, 2, \cdots, k.\right\}$$

with inner product

$$\langle \phi, \psi\rangle = \int_{\mathbb{R}} (\phi'\psi' + \phi\psi)dx.$$

Then problem (3.7) is equivalent to find $\phi \in H$ such that

$$\begin{aligned} \langle \phi, \psi\rangle &= \int_{\mathbb{R}} \Big[\alpha_\varepsilon(p^*+\varepsilon)V^{p^*+\varepsilon-1}\phi + \lambda q\beta_N e^{-(p^*-q)x}V^{q-1}\phi \\ &\qquad + \left(\frac{2}{N-2}\right)^2 e^{-\frac{4}{N-2}x}\phi + h\Big]\psi dx, \end{aligned} \tag{3.21}$$

for all $\psi \in H$. By the Riesz representation theorem, (3.21) is equivalent to solve

$$\phi = K(\phi) + \tilde{h} \tag{3.22}$$

with $\tilde{h} \in H$ depending linearly on h and $K : H \to H$ being a compact operator. Fredholm's alternative yields there is a unique solution to problem (3.22) for any h provided that

$$\phi = K(\phi) \tag{3.23}$$

has only the zero solution in H. (3.23) is equivalent to problem (3.2) with $h = 0$. If $h = 0$, estimate (3.6) implies that $\phi = 0$. This ends the proof of Proposition 3.1.

Now we study the differentiability of the operator T_ε with respect to $\xi = (\xi_1, \cdots, \xi_k)$. Consider the Banach space

$$\mathcal{C}_* = \{f \in C(\mathbb{R}) \ : \ \|f\|_{**} < \infty\}$$

endowed with the $\|\cdot\|_{**}$ norm. The following result holds.

Proposition 3.3. *Under the assumptions of Proposition 3.1, the map $\xi \mapsto T_\varepsilon$ is of class C^1. Moreover,*

$$\|D_\xi T_\varepsilon(h)\|_* \leq C\|h\|_{**}$$

uniformly on the vectors ξ which satisfy (2.10).

Proof. Fix $h \in \mathcal{C}_*$ and let $\phi = T_\varepsilon(h)$ for $\varepsilon < \varepsilon_0$. Let us recall that ϕ satisfies

$$\begin{cases} \mathcal{L}_\varepsilon(\phi) = h + \sum\limits_{j=1}^{k} c_j Z_j \quad \text{ in } (-\infty, +\infty); \\ \lim\limits_{|x|\to\infty} \phi(x) = 0; \\ \int_{\mathbb{R}} Z_j \phi = 0, \quad \forall\, j = 1, \cdots, k, \end{cases}$$

for certain constants c_j. Differentiating above equation with respect to ξ_l, $l \in \{1, \cdots, k\}$. Set $Y = \partial_{\xi_l}\phi$ and $d_j = \partial_{\xi_l} c_j$, we have

$$\begin{cases} \mathcal{L}_\varepsilon(Y) = \overline{h} + \sum\limits_{j=1}^{k} d_j Z_j \quad \text{ in } (-\infty, +\infty); \\ \lim\limits_{|x|\to\infty} Y(x) = 0; \\ \int_{\mathbb{R}} Y Z_j + \phi \partial_{\xi_l} Z_j = 0, \quad \forall\, j = 1, \cdots, k, \end{cases}$$

where

$$\overline{h} = \alpha_\varepsilon(p^*+\varepsilon)(p^*+\varepsilon-1)e^{\varepsilon x}V^{p^*+\varepsilon-2}Z_l\phi + \lambda q(q-1)\beta_N e^{-(p^*-q)x}V^{q-2}Z_l\phi + c_l\partial_{\xi_l}Z_l.$$

Let $\eta = Y - \sum\limits_{i=1}^{k} b_i Z_i$, where $b_i \in \mathbb{R}$ is chosen such that

$$\int_{\mathbb{R}} \eta Z_j = 0,$$

that is,

$$\sum_{i=1}^{k} b_i \int_{\mathbb{R}} Z_i Z_j = \int_{\mathbb{R}} Y Z_j = \int_{\mathbb{R}} \partial_{\xi_l}\phi Z_j = -\int_{\mathbb{R}} \phi \partial_{\xi_l} Z_j. \tag{3.24}$$

This is an almost diagonal system, it has a unique solution and we have

$$|b_i| \leq C\|\phi\|_*. \tag{3.25}$$

Moreover, η satisfies

$$\begin{cases} \mathcal{L}_\varepsilon(\eta) = g + \sum\limits_{j=1}^{k} d_j Z_j \quad \text{ in } (-\infty, +\infty); \\ \lim\limits_{|x|\to\infty} \eta(x) = 0; \\ \int_{\mathbb{R}} \eta Z_j = 0, \quad \forall\, j = 1, \cdots, k, \end{cases} \tag{3.26}$$

with

$$g = \overline{h} - \sum_{i=1}^{k} b_i \mathcal{L}_\varepsilon(Z_i).$$

From Proposition 3.1, there is a unique solution $\eta = T_\varepsilon(g)$ to (3.26) and

$$\|\eta\|_* \leq C\|g\|_{**}. \tag{3.27}$$

Moreover, we have

$$\begin{aligned}\|g\|_{**} &\le C\|e^{\varepsilon x}V^{p^*+\varepsilon-2}Z_l\phi\|_{**}+C\|e^{-(p^*-q)x}V^{q-2}Z_l\phi\|_{**}\\ &\quad +\|c_l\partial_{\xi_l}Z_l\|_{**}+\sum_{i=1}^{k}|b_i|\|\mathcal{L}_\varepsilon(Z_i)\|_{**}\\ &\le C(\|\phi\|_*+|c_l|+|b_i|)\le C\|h\|_{**},\end{aligned}\tag{3.28}$$

because $|b_i|\le C\|\phi\|_*$, $\|\phi\|_*\le C\|h\|_{**}$ and $|c_l|\le C\|h\|_{**}$.

By (3.25), (3.27), (3.28) and $\|Z_i\|_*\le C$, we obtain that

$$\|\partial_{\xi_l}\phi\|_*\le\|\eta\|_*+\sum_{i=1}^{k}|b_i|\|Z_i\|_*\le C\|h\|_{**}.$$

Besides $\partial_{\xi_l}\phi$ depends continuously on ξ in the considered region for this norm. □

4. Nonlinear Problem

In this section, our purpose is to study nonlinear problem. We first have the validity of the following result.

Lemma 4.1. *We have*

$$\|N(\phi)\|_{**}\le C\left(\|\phi\|_*^{\min\{p^*,2\}}+\|\phi\|_*^{\min\{q,2\}}\right),\tag{4.1}$$

and

$$\|\partial_\phi N(\phi)\|_{**}\le C\left(\|\phi\|_*^{\min\{p^*-1,1\}}+\|\phi\|_*^{\min\{q-1,1\}}\right),\tag{4.2}$$

for $\|\phi\|_*\le 1$.

Proof. By the fundamental theorem of calculus and the definition of $\|\ \|_{**}$, we have

$$\begin{aligned}&\|N(\phi)\|_{**}\\ \le\ &\alpha_\varepsilon(p^*+\varepsilon)\sup_{x\in\mathbb{R}}\left(\sum_{j=1}^{k}e^{-\sigma|x-\xi_j|}\right)^{-1}e^{\varepsilon x}\left|\int_0^1\left[(V+t\phi)^{p^*+\varepsilon-1}-V^{p^*+\varepsilon-1}\right]\phi\ dt\right|\\ &+\lambda q\beta_N\sup_{x\in\mathbb{R}}\left(\sum_{j=1}^{k}e^{-\sigma|x-\xi_j|}\right)^{-1}e^{-(p^*-q)x}\left|\int_0^1\left[(V+t\phi)^{q-1}-V^{q-1}\right]\phi\ dt\right|\\ :=\ &N_1+N_2.\end{aligned}$$

Using

$$||a+b|^q-|a|^q|\le C\begin{cases}|a|^{q-1}|b|+|b|^q & \text{if } q\ge1;\\ \min\{|a|^{q-1}|b|,|b|^q\} & \text{if } 0<q<1,\end{cases}$$

if $p^*\ge2$ and for $\|\phi\|_*\le1$, we have

$$\begin{aligned}N_1 &\le C\sup_{x\in\mathbb{R}}\left(\sum_{j=1}^{k}e^{-\sigma|x-\xi_j|}\right)^{-1}e^{\varepsilon x}V^{p^*+\varepsilon-2}|\phi|^2+C\sup_{x\in\mathbb{R}}\left(\sum_{j=1}^{k}e^{-\sigma|x-\xi_j|}\right)^{-1}e^{\varepsilon x}|\phi|^{p^*+\varepsilon}\\ &\le C\|\phi\|_*^2+C\|\phi\|_*^{p^*+\varepsilon}\le C\|\phi\|_*^2.\end{aligned}$$

Similarly, if $1 < p^* < 2$, we find that $N_1 \le C\|\phi\|_*^{p^*}$. Thus we get

$$N_1 \le C\|\phi\|_*^{\min\{p^*,2\}}.$$

Moreover, by similar computations as N_1, we can conclude that

$$N_2 \le C\|\phi\|_*^{\min\{q,2\}}.$$

Thus we get (4.1).

We differentiate $N(\phi)$ with respect to ϕ, we have

$$\partial_\phi N(\phi) = \alpha_\varepsilon (p^*+\varepsilon) e^{\varepsilon x}\left[(V+\phi)^{p^*+\varepsilon-1} - V^{p^*+\varepsilon-1}\right] + \lambda\beta_N q e^{-(p^*-q)x}\left[(V+\phi)^{q-1} - V^{q-1}\right].$$

By a similar argument as $\|N(\phi)\|_{**}$, (4.2) holds. □

Lemma 4.2. *Let $\sigma > 0$ satisfy (3.3) and $0 < \xi_1 < \xi_2 < \cdots < \xi_k$ satisfy (2.10). If q satisfies (1.4), then there exist $\tau \in (\frac{1}{2}, 1)$ and a constant $C > 0$, such that*

$$\|E\|_{**} \le C\varepsilon^\tau, \qquad \|\partial_\xi E\|_{**} \le C\varepsilon^\tau.$$

Proof. We have

$$\begin{aligned} E &= \alpha_\varepsilon e^{\varepsilon x}\left(V^{p^*+\varepsilon} - V^{p^*}\right) + (\alpha_\varepsilon e^{\varepsilon x} - 1)V^{p^*} + \left(V^{p^*} - \left(\sum_{j=1}^k W_j\right)^{p^*}\right) \\ &\quad + \left(\left(\sum_{j=1}^k W_j\right)^{p^*} - \sum_{j=1}^k W_j^{p^*}\right) + \lambda\beta_N e^{-(p^*-q)x} V^q \\ &:= E_1 + E_2 + E_3 + E_4 + E_5. \end{aligned} \tag{4.3}$$

Estimate of E_1:

$$|E_1| = \left|\varepsilon\alpha_\varepsilon e^{\varepsilon x}\int_0^1 V^{p^*+t\varepsilon}\log V\,dt\right| \le C\varepsilon\sum_{j=1}^k e^{-\sigma|x-\xi_j|}.$$

Estimate of E_2: By the Taylor expansion, we have

$$\begin{aligned} |E_2| &= \left|\left(\left(\frac{p^*-1}{2}\right)^{-\frac{2\varepsilon}{p^*-1}} e^{\varepsilon x} - 1\right)V^{p^*}\right| \\ &= \left(\varepsilon x\int_0^1 e^{t\varepsilon x}\,dt + O(\varepsilon)e^{\varepsilon x}\right)V^{p^*} \le C\varepsilon|\log\varepsilon|\sum_{j=1}^k e^{-\sigma|x-\xi_j|}. \end{aligned}$$

Estimate of E_3: Since

$$|E_3| = \left|V^{p^*} - \left(\sum_{j=1}^k W_j\right)^{p^*}\right| \le CV^{p^*-1}\sum_{j=1}^k |R_{\xi_j}(x)|.$$

Thanks to Lemma 2.2, for $x \le 0$, we have

$$|E_3| \le CV^{p^*-1}\sum_{j=1}^k e^{-|x-\xi_j|} \le CV^{p^*-1}e^{-\xi_1} \le C\varepsilon^{\frac{1}{p^*-q}}\sum_{j=1}^k e^{-\sigma|x-\xi_j|}.$$

For $0 \le x \le \xi_1$,

$$\begin{aligned} |E_3| &\le CV^{p^*-1}\sum_{j=1}^k e^{-|x-\xi_j|}e^{-\frac{2}{N-2}\min\{x,\xi_j\}} \\ &\le C\sum_{j=1}^k e^{-\sigma|x-\xi_j|}\begin{cases} \varepsilon^{\frac{2}{N+2-(N-2)q}} & \text{if } N\ge 4; \\ \varepsilon^{\frac{1}{5-q}} & \text{if } N=3. \end{cases} \end{aligned}$$

If $x \ge \xi_1$, for $0<\sigma<p^*-1$, we have

$$\begin{aligned} |E_3| &\le CV^{p^*-1}\sum_{j=1}^k e^{-|x-\xi_j|}e^{-\frac{2}{N-2}\min\{x,\xi_j\}} \\ &\le CV^{p^*-1}e^{-\frac{2}{N-2}\xi_1} \le C\varepsilon^{\frac{2}{N+2-(N-2)q}}\sum_{j=1}^k e^{-\sigma|x-\xi_j|}. \end{aligned}$$

Therefore we get for $x \in \mathbb{R}$,

$$|E_3| \le C\sum_{j=1}^k e^{-\sigma|x-\xi_j|}\begin{cases} \varepsilon^{\frac{2}{N+2-(N-2)q}} & \text{if } N\ge 4; \\ \varepsilon^{\frac{1}{5-q}} & \text{if } N=3. \end{cases}$$

Estimate of E_4: If $-\infty < x \le \frac{\xi_1+\xi_2}{2}$, we have

$$\begin{aligned} |E_4| &\le \left|\left(\sum_{j=1}^k W(x-\xi_j)\right)^{p^*} - W(x-\xi_1)^{p^*}\right| + \left|\sum_{j=2}^k W(x-\xi_j)^{p^*}\right| \\ &\le p^*\left(\sum_{j=1}^k W(x-\xi_j)\right)^{p^*-1}\sum_{j=2}^k W(x-\xi_j) + \sum_{j=2}^k W(x-\xi_j)^{p^*} \\ &= p^*\left(\sum_{j=1}^k W(x-\xi_j)\right)^{p^*-1-\theta}\left(\sum_{j=1}^k W(x-\xi_j)\right)^{\theta}\sum_{j=2}^k W(x-\xi_j) + \sum_{j=2}^k W(x-\xi_j)^{p^*} \end{aligned}$$

with a positive number θ, satisfying $0<\theta<p^*-1-\sigma$. Note that

$$\left(\sum_{j=1}^k W(x-\xi_j)\right)^{\theta}\sum_{j=2}^k W(x-\xi_j) \le C\varepsilon^{\frac{1+\theta}{2}}.$$

Moreover,

$$\sum_{j=2}^k W(x-\xi_j)^{p^*} \le C\varepsilon^{\frac{p^*-\sigma}{2}}\sum_{j=1}^k e^{-\sigma|x-\xi_j|}.$$

Thus

$$|E_4| \le C\varepsilon^{\frac{1+\theta}{2}}\sum_{j=1}^k e^{-\sigma|x-\xi_j|}, \quad \text{for} \quad -\infty < x \le \frac{\xi_1+\xi_2}{2},$$

with $0<\theta<p^*-1-\sigma$. Similarly, for $\frac{\xi_{l-1}+\xi_l}{2}\le x\le \frac{\xi_l+\xi_{l+1}}{2}$ with $l=2,\cdots,k-1$, and $x\ge \frac{\xi_{k-1}+\xi_k}{2}$, we get

$$|E_4|\le C\varepsilon^{\frac{1+\theta}{2}}\sum_{j=1}^k e^{-\sigma|x-\xi_j|}.$$

Therefore for $x\in\mathbb{R}$, we have

$$|E_4|\le C\varepsilon^{\frac{1+\theta}{2}}\sum_{j=1}^k e^{-\sigma|x-\xi_j|},\quad \text{where } 0<\theta<p^*-1-\sigma.$$

The estimate of E_5 is similar as the previous ones and we get

$$|E_5|\le C\max\{\varepsilon,\varepsilon^{\frac{q-\sigma}{p^*-q}}\}\sum_{j=1}^k e^{-\sigma|x-\xi_j|}.$$

From (4.3) and the previous estimates, for $0<\theta<p^*-1-\sigma$ with σ satisfying (3.3), we have

$$\|E\|_{**}\le C\begin{cases}\max\left\{\varepsilon|\log\varepsilon|,\ \varepsilon^{\frac{2}{N+2-(N-2)q}},\ \varepsilon^{\frac{1+\theta}{2}},\varepsilon^{\frac{q-\sigma}{p^*-q}}\right\} & \text{if } N\ge 4;\\ \max\left\{\varepsilon|\log\varepsilon|,\varepsilon^{\frac{1}{5-q}},\varepsilon^{\frac{1+\theta}{2}},\varepsilon^{\frac{q-\sigma}{p^*-q}}\right\} & \text{if } N=3.\end{cases}$$

Therefore if q satisfies (1.4), we find that there esists $\tau\in(\frac12,1)$ such that

$$\|E\|_{**}\le C\varepsilon^\tau.$$

Differentiating E with respect to ξ_i $(i=1,2,\cdots,k)$, we have

$$\begin{aligned}\partial_{\xi_i}E &= \alpha_\varepsilon(p^*+\varepsilon)e^{\varepsilon x}V^{p^*+\varepsilon-1}\partial_{\xi_i}V-p^*\sum_{j=1}^k W(x-\xi_j)^{p^*-1}\partial_{\xi_i}W(x-\xi_j)\\ &\quad +\lambda\beta_N q e^{-(p^*-q)x}V^{q-1}\partial_{\xi_i}V.\end{aligned}$$

The proof of estimate for $\|\partial_\xi E\|_{**}$ is similar to that of $\|E\|_{**}$. $\square$

Proposition 4.3. *Assume that $0<\xi_1<\xi_2<\cdots<\xi_k$ satisfy (2.10). Then there exists $C>0$ such that for $\varepsilon>0$ small enough, there exists a unique solution $\phi=\phi(\xi)$ to problem (3.1) with*

$$\|\phi\|_*\le C\varepsilon^\tau,$$

for some $\tau\in(\frac12,1)$ satisfying Lemma 4.2. Moreover, the map $\xi\mapsto\phi(\xi)$ is of class C^1 for the $\|\cdot\|_*$ norm, and

$$\|\partial_\xi\phi\|_*\le C\varepsilon^\tau.$$

Proof. Problem (3.1) is equivalent to solve a fixed point problem

$$\phi=T_\varepsilon(N(\phi)+E):=A_\varepsilon(\phi).$$

We will show that the operator A_ε is a contraction map in a proper region. Set

$$\mathcal{F}_\gamma=\{\phi\in C(\mathbb{R}):\|\phi\|_*\le\gamma\varepsilon^\tau\},$$

where $\gamma>0$ will be chosen later.

For $\phi\in\mathcal{F}_\gamma$, by Lemmas 4.1 and 4.2, we get

$$\|A_\varepsilon(\phi)\|_* = \|T_\varepsilon(N(\phi)+E)\|_*\le C\|N(\phi)\|_{**}+\|E\|_{**}$$

$$\leq \quad C\left(\gamma^{\min\{p^*,2\}}\varepsilon^{\min\{p^*-1,1\}\tau} + \gamma^{\min\{q,2\}}\varepsilon^{\min\{q-1,1\}\tau} + 1\right)\varepsilon^{\tau}.$$

Then we have $A_\varepsilon(\phi) \in \mathcal{F}_\gamma$ for $\phi \in \mathcal{F}_\gamma$ by choosing γ large enough but fixed.

Moreover, for $\phi_1, \phi_2 \in \mathcal{F}_\gamma$, by writing

$$N(\phi_1) - N(\phi_2) = \int_0^1 N'(\phi_2 + t(\phi_1 - \phi_2))dt(\phi_1 - \phi_2).$$

By Proposition 3.1 and using (4.2), we find

$$\begin{aligned} &\|A_\varepsilon(\phi_1) - A_\varepsilon(\phi_2)\|_* \leq C\|N(\phi_1) - N(\phi_2)\|_{**} \\ \leq \quad & C\left(\left(\max_{i=1,2}\|\phi_i\|_*\right)^{\min\{p^*-1,1\}} + \left(\max_{i=1,2}\|\phi_i\|_*\right)^{\min\{q-1,1\}}\right)\|\phi_1 - \phi_2\|_* \\ \leq \quad & C\varepsilon^\kappa\|\phi_1 - \phi_2\|_* \end{aligned}$$

with some $\kappa > 0$. This implies that A_ε is a contraction map from $\mathcal{F}_\gamma$ to $\mathcal{F}_\gamma$. Thus A_ε has a unique fixed point in $\mathcal{F}_\gamma$.

Now we consider the differentiability of $\xi \mapsto \phi(\xi)$. We write

$$B(\xi, \phi) := \phi - T_\varepsilon(N(\phi) + E).$$

First we observe that $B(\xi, \phi) = 0$. Moreover,

$$\partial_\phi B(\xi, \phi)[\theta] = \theta - T_\varepsilon(\theta(\partial_\phi(N(\phi)))) \equiv \theta + M(\theta),$$

where

$$M(\theta) = -T_\varepsilon(\theta(\partial_\phi(N(\phi)))).$$

By a direct calculation, we get

$$\|M(\theta)\|_* \leq C\|\theta(\partial_\phi(N(\phi)))\|_{**} \leq C\varepsilon^\kappa\|\theta\|_*.$$

So for $\varepsilon > 0$ small enough, the operator $\partial_\phi B(\xi, \phi)$ is invertible with uniformly bounded inverse in $\|\cdot\|_*$. It also depends continuously on its parameters. Let us differentiate with respect to ξ, we have

$$\partial_\xi B(\xi, \phi) = -(\partial_\xi T_\varepsilon)(N(\phi) + E) - T_\varepsilon((\partial_\xi N)(\xi, \phi) + \partial_\xi E),$$

where all these expressions depend continuously on their parameters. The implicit function theorem yields that $\phi(\xi)$ is of class C^1 and

$$\partial_\xi \phi = -(\partial_\phi B(\xi, \phi))^{-1}[\partial_\xi B(\xi, \phi)]$$

so that

$$\|\partial_\xi \phi\|_* \leq C\left(\|N(\phi)\|_{**} + \|E\|_{**} + \|(\partial_\xi N)(\xi, \phi)\|_{**} + \|\partial_\xi E\|_{**}\right) \leq C\varepsilon^\tau.$$

□

5. The finite-dimensional variational reduction

According to the results of the previous section, our problem has been reduced to find points $\xi = (\xi_1, \xi_2, \cdots, \xi_k)$, such that

$$c_j(\xi) = 0 \qquad \text{for all } j = 1, \cdots, k. \tag{5.1}$$

If (5.1) holds, then $v = V + \phi$ is a solution to (1.7), and $u = \sum_{j=1}^k U_{\mu_j} + \psi$ is the solution to problem (1.3) with $\psi = \mathcal{T}^{-1}(\phi)$.

Define the function $\mathcal{I}_\varepsilon : (\mathbb{R}^+)^k \to \mathbb{R}$ as

$$\mathcal{I}_\varepsilon(\xi) := I_\varepsilon(V + \phi),$$

where V is defined by (2.11) and I_ε is the energy functional of (1.7) defined by

$$\begin{aligned}I_\varepsilon(v) &= \frac{1}{2}\int_{-\infty}^{+\infty}(|v'(x)|^2+|v|^2)dx+\frac{1}{2}\left(\frac{2}{N-2}\right)^2\int_{-\infty}^{+\infty}e^{-\frac{4}{N-2}x}v^2dx\\&\quad-\frac{1}{p^*+\varepsilon+1}\alpha_\varepsilon\int_{-\infty}^{+\infty}e^{\varepsilon x}|v|^{p^*+\varepsilon+1}dx-\frac{1}{q+1}\lambda\beta_N\int_{-\infty}^{+\infty}e^{-(p^*-q)x}|v|^{q+1}dx.\end{aligned}$$

We have the following fact.

Lemma 5.1. *The function $V+\phi$ is a solution to* (1.7) *if and only if $\xi=(\xi_1,\cdots,\xi_k)$ is a critical point of $\mathcal{I}_\varepsilon(\xi)$, where $\phi=\phi(\xi)$ is given by Proposition 4.3.*

Proof. For $s\in\{1,2,\cdots,k\}$, we have

$$\begin{aligned}\partial_{\xi_s}\mathcal{I}_\varepsilon(\xi) &= \partial_{\xi_s}(I_\varepsilon(V+\phi))=DI_\varepsilon(V+\phi)[\partial_{\xi_s}V+\partial_{\xi_s}\phi]\\&= \sum_{j=1}^k c_j\int_{\mathbb{R}}Z_j[\partial_{\xi_s}V+\partial_{\xi_s}\phi]=\sum_{j=1}^k c_j\left(\int_{\mathbb{R}}Z_jZ_sdx+o(1)\right),\end{aligned}$$

where $o(1)\to0$ as $\varepsilon\to0$ uniformly for the norm $\|\cdot\|_*$. This implies that the above relations define an almost diagonal homogeneous linear equation system for the c_j. Thus ξ is the critical point of I_ε if and only if $c_j=0$ for all $j=1,2,\cdots,k$. □

Lemma 5.2. *The following expansion holds*

$$\mathcal{I}_\varepsilon(\xi)=I_\varepsilon(V)+o(\varepsilon)$$

as $\varepsilon\to0$, where $o(\varepsilon)$ is uniform in the C^1-sense on the vectors ξ satisfying (2.10).

Proof. By the fact that $DI_\varepsilon(V+\phi)[\phi]=0$ and using the Taylor expansion, we have

$$\begin{aligned}&\mathcal{I}_\varepsilon(\xi)-I_\varepsilon(V)=I_\varepsilon(V+\phi)-I_\varepsilon(V)=\int_0^1D^2I_\varepsilon(V+t\phi)[\phi^2]tdt\\&=\int_0^1tdt\int_{-\infty}^{+\infty}(N(\phi)+E)\phi dx\\&\quad+(p^*+\varepsilon)\alpha_\varepsilon\int_0^1tdt\int_{-\infty}^{+\infty}e^{\varepsilon x}\left[V^{p^*+\varepsilon-1}-(V+t\phi)^{p^*+\varepsilon-1}\right]\phi^2dx\\&\quad+\lambda\beta_Nq\int_0^1tdt\int_{-\infty}^{+\infty}e^{-(p^*-q)x}\left[V^{q-1}-(V+t\phi)^{q-1}\right]\phi^2dx.\end{aligned}$$

Since $\|\phi\|_*\le C\varepsilon^\tau$ and $\|E\|_{**}\le C\varepsilon^\tau$ with $\tau>\frac{1}{2}$, we get

$$\mathcal{I}_\varepsilon(\xi)-I_\varepsilon(V)=O(\varepsilon^{2\tau})=o(\varepsilon)$$

uniformly on the points ξ which satisfy (2.10).

Moreover, differentiating with respect to ξ_s, we have

$$\begin{aligned}&\partial_{\xi_s}\left(\mathcal{I}_\varepsilon(\xi)-I_\varepsilon(V)\right)=\int_0^1\int_{-\infty}^{+\infty}\partial_{\xi_s}[(N(\phi)+E)\phi]tdxdt\\&+\alpha_\varepsilon(p^*+\varepsilon)\int_0^1tdt\int_{-\infty}^{+\infty}e^{\varepsilon x}\partial_{\xi_s}\left(\left[V^{p^*+\varepsilon-1}-(V+t\phi)^{p^*+\varepsilon-1}\right]\phi^2\right)dx\\&+\lambda\beta_Nq\int_0^1tdt\int_{-\infty}^{+\infty}e^{-(p^*-q)x}\partial_{\xi_s}\left(\left[V^{q-1}-(V+t\phi)^{q-1}\right]\phi^2\right)dx.\end{aligned}$$

By the fact that $\|\partial_\xi \phi\|_* \le C\varepsilon^\tau$ and $\|\partial_\xi E\|_{**} \le C\varepsilon^\tau$ with $\tau > \frac{1}{2}$, we deduce that

$$\partial_{\xi_s}\left(\mathcal{I}_\varepsilon(\xi) - I_\varepsilon(V)\right) = O(\varepsilon^{2\tau}) = o(\varepsilon).$$

□

Now we consider the energy functional of problem (1.3), which is defined by

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) - \frac{1}{p^*+1+\varepsilon}\int_{\mathbb{R}^N} |u|^{p^*+1+\varepsilon} - \frac{\lambda}{q+1}\int_{\mathbb{R}^N} |u|^{q+1}.$$

By a direct calculation, we have that

$$I_\varepsilon(V) = \left(\frac{2}{N-1}\right)^{N-1} \frac{1}{\omega_{N-1}} J(U), \tag{5.2}$$

where V is defined by (2.11), ω_{N-1} is the volume of the unit sphere in $\mathbb{R}^N$ and $U(z) = \sum_{j=1}^k U_{\mu_j}(z)$ with U_{μ_j} satisfying problem (2.1).

We give the following expansion of $J(U)$, whose proof is in the Appendix.

Lemma 5.3. *Assume that (2.8) and (2.9) hold, then we have the following expansion:*

$$J(U) = a_1 + a_2\varepsilon - \varphi(\Lambda_1, \cdots, \Lambda_k)\varepsilon + a_3 \varepsilon \log\varepsilon + o(\varepsilon), \tag{5.3}$$

where

$$\varphi(\Lambda_1, \cdots, \Lambda_k) = a_4 \Lambda_1^{\frac{N+2-(N-2)q}{2}} - a_5 \sum_{i=1}^k \log\Lambda_i + a_6 \sum_{l=1}^{k-1}\left(\frac{\Lambda_{l+1}}{\Lambda_l}\right)^{\frac{N-2}{2}}, \tag{5.4}$$

and as $\varepsilon \to 0$, $o(\varepsilon)$ is uniform in the C^1-sense on the Λ_i's satisfying (2.8), and

$$\begin{aligned}
a_1 &= \frac{k}{N}\alpha_N^{p^*+1}\int_{\mathbb{R}^N} \frac{1}{(1+|z|^2)^N}dz,\\
a_2 &= \frac{k}{(p^*+1)^2}\alpha_N^{p^*+1}\int_{\mathbb{R}^N} \frac{1}{(1+|z|^2)^N}dz\\
&\quad - \frac{k}{p^*+1}\alpha_N^{p^*+1}\int_{\mathbb{R}^N} \frac{1}{(1+|z|^2)^N}\log\frac{\alpha_N}{(1+|z|^2)^{\frac{N-2}{2}}}dz,\\
a_3 &= \frac{(N-2)^2}{4N}\left(\alpha_N^{p^*+1}\int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^N}dz\right)\\
&\quad \times \sum_{i=1}^k\left(\frac{2(i-1)}{N-2} + \frac{2}{N+2-(N-2)q}\right),\\
a_4 &= \frac{\lambda}{q+1}\int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^{\frac{(N-2)(q+1)}{2}}}dz,\\
a_5 &= \frac{(N-2)^2}{4N}\left(\alpha_N^{p^*+1}\int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^N}dz\right),\\
a_6 &= \alpha_N^{p^*+1}\int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^{\frac{N+2}{2}}}\frac{1}{|z|^{N-2}}dz.
\end{aligned}$$

Now we are ready to prove our main result.

Proof of Theorem 1.1. Thanks to Lemma 5.1, we know that

$$u = \sum_{j=1}^{k} U_{\mu_j} + \psi \quad \text{with } \psi = \mathcal{T}^{-1}(\phi)$$

is a solution to problem (1.3) if and only if ξ is a critical point of $\mathcal{I}_\varepsilon(\xi)$, where the existence of ϕ is guaranteed by Proposition 4.3.

Finding a critical point of $\mathcal{I}_\varepsilon(\xi)$ is equivalent to find that of $\widetilde{\mathcal{I}}_\varepsilon(\xi)$, which is defined as

$$\widetilde{\mathcal{I}}_\varepsilon(\xi) = -\left(\frac{N-1}{2}\right)^{N-1} \frac{\omega_{N-1}}{\varepsilon}\mathcal{I}_\varepsilon(\xi) + \frac{a_1}{\varepsilon} + a_2 + a_3 \log \varepsilon.$$

On the other hand, from Lemmas 5.2 and 5.3, using (5.2), we have

$$\begin{aligned}
\mathcal{I}_\varepsilon(\xi) &= I_\varepsilon(V) + o(\varepsilon) = \left(\frac{2}{N-1}\right)^{N-1} \frac{1}{\omega_{N-1}} J(U) + o(\varepsilon) \\
&= \left(\frac{2}{N-1}\right)^{N-1} \frac{1}{\omega_{N-1}} \left[a_1 + a_2\varepsilon - \varphi(\Lambda_1, \cdots, \Lambda_k)\varepsilon + a_3 \varepsilon \log \varepsilon\right] + o(\varepsilon),
\end{aligned}$$

as $\varepsilon \to 0$, where $\varphi(\Lambda)$ is defined by (5.4) and $o(\varepsilon)$ is uniform in the C^1-sense. Then we have

$$\widetilde{\mathcal{I}}_\varepsilon(\xi) = \varphi(\Lambda) + o(1), \tag{5.5}$$

where $o(1)$ is uniform in the C^1-sense as $\varepsilon \to 0$.

We set $s_1 = \Lambda_1$, $s_j = \frac{\Lambda_j}{\Lambda_{j-1}}$, then we can write $\varphi(\Lambda_1, \cdots, \Lambda_k)$ as

$$\begin{aligned}
\varphi(s_1, \cdots, s_k) &= a_4 s_1^{\frac{N+2-(N-2)q}{2}} - a_5 k \log s_1 - \sum_{j=2}^{k} \left[a_5(k-j+1)\log s_j - a_6 s_j^{\frac{N-2}{2}}\right] \\
&:= \tilde{\varphi}_1 - \sum_{j=2}^{k} \tilde{\varphi}_j,
\end{aligned}$$

with

$$\tilde{\varphi}_1 = a_4 s_1^{\frac{N+2-(N-2)q}{2}} - a_5 k \log s_1$$

and

$$\tilde{\varphi}_j = a_5(k-j+1)\log s_j - a_6 s_j^{\frac{N-2}{2}}, \quad j = 2, \cdots, k.$$

We note that

$$\bar{s}_1 = \left(\frac{2a_5 k}{a_4(N+2-(N-2)q)}\right)^{\frac{2}{N+2-(N-2)q}} \tag{5.6}$$

is the critical point of $\tilde{\varphi}_1$, and

$$\bar{s}_j = \left(\frac{2a_5(k-j+1)}{(N-2)a_6}\right)^{\frac{2}{N-2}}, \qquad j = 2, \cdots, k, \tag{5.7}$$

is the critical point of $\tilde{\varphi}_j$. Moreover

$$\tilde{\varphi}_1''(\bar{s}_1) < 0, \quad \tilde{\varphi}_j''(\bar{s}_j) < 0, \quad j = 2, \cdots, k.$$

So $(\bar{s}_1, \bar{s}_2, \cdots, \bar{s}_k)$ is a nondegenerate critical point of $\varphi(s_1, \cdots, s_k)$. Thus

$$\Lambda^* := (\bar{s}_1, \bar{s}_2\bar{s}_1, \bar{s}_3\bar{s}_2\bar{s}_1, \cdots, \bar{s}_k \times \cdots \times \bar{s}_2\bar{s}_1)$$

is a nondegenerate critical point of $\varphi(\Lambda)$. It follows that the local degree $deg(\nabla\varphi(\Lambda), \mathcal{O}, 0)$ is well defined and is nonzero, here $\mathcal{O}$ is an arbitrarily small neighborhood of Λ^*. Hence from (5.5), for $\varepsilon > 0$ small enough, we have that $deg(\nabla_\xi \widetilde{\mathcal{I}}_\varepsilon(\xi), \bar{\mathcal{O}}, 0) \neq 0$, where $\bar{\mathcal{O}}$ is a small neighborhood of $\xi^* = (\xi_1^*, \cdots, \xi_k^*)$ and

$$\xi_j^* = \left[(j-1) + \frac{1}{p^* - q}\right] \log\frac{1}{\varepsilon} - \frac{N-2}{2}\log\left(\bar{s}_j \bar{s}_{j-1}\cdots\bar{s}_1\right), \text{ for } \forall\ j = 1, \cdots, k.$$

So ξ^* is a critical point of $\widetilde{\mathcal{I}}_\varepsilon(\xi)$, which implies there is a critical point of $\mathcal{I}_\varepsilon$.

Furthermore, if for some i, $|x - \xi_i| \leq C_0$ with some $C_0 > 0$, then we have $|\phi| = o(W(x - \xi_i))$. Thus $\psi(|z|) = \mathcal{T}^{-1}(\phi(x)) = o(w_{\mu_i})$ for $\frac{1}{C}\mu_i \leq |z| \leq C\mu_i$. Moreover, from (c) of Lemma 2.1, we get that $R_{\mu_i} = o(w_{\mu_i})$ for $\frac{1}{C}\mu_i \leq |z| \leq C\mu_i$. Therefore we obtain (1.5) holds with

$$\Lambda_j^* = \bar{s}_j \bar{s}_{j-1}\cdots\bar{s}_1, \quad j = 1, \cdots, k,$$

where $\bar{s}_j$ are given by (5.6) and (5.7). This finishes the proof. □

6. Appendix

6.1. **Proof of Lemma 2.1.** In order to prove Lemma 2.1, we introduce the Green function. For a fixed $z \in \mathbb{R}^N$, let $G(z, y)$ be the Green function of $-\Delta + Id$, which satisfies

$$\begin{aligned} -\Delta G(z,y) + G(z,y) &= \delta_z(y) \quad \text{in } \mathbb{R}^N, \\ G(z,y) \to 0 & \qquad |y| \to \infty. \end{aligned}$$

We have the following result.

Lemma 6.1. *We have*

$$|G(z,y)| \leq \frac{C}{|y-z|^{N-2}} \qquad \textit{for } \ 0 < |y-z| \leq 1,$$

and

$$|G(z,y)| \leq C|y-z|^{\frac{1-N}{2}} e^{-|y-z|} \qquad \textit{for } \ |y-z| \geq 1.$$

Proof. By radial symmetry, we can write $G(z, y) = G(r)$ with $r = |y - z|$. Since $G(r)$ is singular at zero and tends to zero at infinity, we can verify that G is given by

$$G(r) = \frac{N-2}{(2\pi)^{\frac{N}{2}}\Gamma(\frac{N}{2})^2} r^{\frac{2-N}{2}} K_{\frac{N-2}{2}}(r),$$

where $K_{\frac{N-2}{2}}(r)$ is a Modified Bessel Function of the Second Kind, see [15]. For $N = 3$, the function G has the explicit form $G(r) = \frac{e^{-r}}{4\pi r}$. In general, we have that $K_{\frac{N-2}{2}}(r) \sim \frac{\Gamma(\frac{N-2}{2})}{2}(\frac{2}{r})^{\frac{N-2}{2}}$ for r close to 0, and $K_{\frac{N-2}{2}}(r) \sim \sqrt{\frac{\pi}{2r}}e^{-r}$ for r large. Using these estimates, we obtain the result. □

Proof of Lemma 2.1. (a) It is a direct consequence of the maximum principle.

(b) Define the barrier function $Q(z) = \mu^{\frac{N-2}{2}}|z|^{-(N+2)}$. It satisfies $-\Delta Q(z) + Q(z) \geq c\mu^{\frac{N-2}{2}}|z|^{-(N+2)}$ for all $|z| \geq R$ with $R > 0$ a large constant, here c is positive constant. Since $Q(z) = \mu^{\frac{N-2}{2}}R^{-(N+2)}$ for $|z| = R$ and $U_\mu(z) \leq w_\mu(z) \leq \alpha_N \mu^{\frac{N-2}{2}}|z|^{-(N-2)}$ for all $|z| \geq 0$. Set $\varphi(z) = AQ(z) - U_\mu(z)$ for some constant $A > 0$, we then have $-\Delta\varphi(z) + \varphi(z) \geq 0$ for $|z| \geq R$, and $\varphi(z) \geq 0$ for $|z| = R$ by choosing suitable constant A. By the maximum principle we get $U_\mu(z) \le AQ(z) = A\mu^{\frac{N-2}{2}}|z|^{-(N+2)}$ for $|z| \ge R$.

(c) Using the representation

$$R_\mu(z) = \int_{\mathbb{R}^N} G(y-z) w_\mu(y) dy$$

and standard convolution estimates we can obtain the stated bounds for R_μ. □

Set

$$\widetilde{Z}_\mu(z) = \partial_\mu U_\mu(z), \qquad \overline{Z}_\mu(z) = \partial_\mu w_\mu(z),$$

then $\tilde{Z}_\mu(z)$ satisfies

$$\begin{cases} -\Delta \widetilde{Z}_\mu + \widetilde{Z}_\mu = \frac{N+2}{N-2} w_\mu^{\frac{4}{N-2}} \overline{Z}_\mu \quad \text{in } \mathbb{R}^N, \\ \widetilde{Z}_\mu(z) \to 0 \ \text{ as } \ |z| \to \infty. \end{cases}$$

We can write

$$\widetilde{Z}_\mu(z) = \overline{Z}_\mu(z) + \partial_\mu R_\mu(z),$$

then $\partial_\mu R_\mu(z)$ satisfies

$$\begin{cases} -\Delta(\partial_\mu R_\mu(z)) + \partial_\mu R_\mu(z) = -\partial_\mu w_\mu(z) \quad \text{in } \mathbb{R}^N, \\ \partial_\mu R_\mu(z) \to 0 \ \text{ as } \ |z| \to \infty. \end{cases}$$

We observe that $|-\partial_\mu w_\mu(z)| \le C\mu^{-1} w_\mu$, then we have

Corollary 6.2. *One has*

$$|\partial_\mu R_\mu(z)| \le C\mu^{-1} |R_\mu(z)| \quad \text{for } \forall\ z \in \mathbb{R}^N. \tag{6.1}$$

Moreover, by the maximum principle, we have that

$$|\widetilde{Z}_\mu(z)| \le C\mu^{\frac{N-4}{2}} |z|^{-(N+2)} \qquad \text{for } \ |z| \ge R, \tag{6.2}$$

where R is a large positive number but fixed in Lemma 2.1.

6.2. **Expansion of energy.**

Proof of Lemma 5.3. The proof is very similar to the one in [20]. The difference is that we have more terms in the energy and the initial approximation is also somewhat different. We have

$$\begin{aligned} J(U) &= \left[\frac{1}{2}\int_{\mathbb{R}^N} (|\nabla U|^2 + U^2) - \frac{1}{p^*+1}\int_{\mathbb{R}^N} U^{p^*+1}\right] \\ &\quad + \left[\frac{1}{p^*+1}\int_{\mathbb{R}^N} U^{p^*+1} - \frac{1}{p^*+1+\varepsilon}\int_{\mathbb{R}^N} U^{p^*+1+\varepsilon}\right] - \frac{\lambda}{q+1}\int_{\mathbb{R}^N} U^{q+1} \\ &:= J_1 + J_2 + J_3, \end{aligned} \tag{6.3}$$

where $U = \sum_{j=1}^{k} U_{\mu_j}$ with $U_{\mu_j} = w_{\mu_j} + R_{\mu_j}$.

As in [20] but using the estimates of R_μ in Lemma 2.1 we can get

$$\begin{aligned} J_1 &= \frac{k}{N}\alpha_N^{p^*+1}\int_{\mathbb{R}^N} \frac{1}{(1+|z|^2)^N} dz \\ &\quad -\varepsilon \sum_{l=1}^{k-1} \left(\frac{\Lambda_{l+1}}{\Lambda_l}\right)^{\frac{N-2}{2}} \alpha_N^{p^*+1}\int_{\mathbb{R}^N} \frac{1}{(1+|z|^2)^{\frac{N+2}{2}}}\frac{1}{|z|^{N-2}} dz + o(\varepsilon). \end{aligned} \tag{6.4}$$

Also as in [20] we obtain

$$
\begin{aligned}
J_2 \quad = \quad & \varepsilon \frac{k}{(p^*+1)^2} \alpha_N^{p^*+1} \int_{\mathbb{R}^N} \frac{1}{(1+|z|^2)^N} dz \\
& -\varepsilon \frac{k}{p^*+1} \alpha_N^{p^*+1} \int_{\mathbb{R}^N} \frac{1}{(1+|z|^2)^N} \log \frac{\alpha_N}{(1+|z|^2)^{\frac{N-2}{2}}} dz \\
& +\varepsilon \frac{(N-2)^2}{4N} \left(\alpha_N^{p^*+1} \int_{\mathbb{R}^N} \frac{1}{(1+|z|^2)^N} dz \right) \sum_{i=1}^k \log \Lambda_i \\
& + \frac{(N-2)^2}{4N} \left(\alpha_N^{p^*+1} \int_{\mathbb{R}^N} \frac{1}{(1+|z|^2)^N} dz \right) \\
& \times \sum_{i=1}^k \left(\frac{2(i-1)}{N-2} + \frac{2}{N+2-(N-2)q} \right) \varepsilon \log \varepsilon + o(\varepsilon). \qquad (6.5)
\end{aligned}
$$

We will do with detail the estimate of the term J_3.

Given $\delta > 0$ small but fixed. Let $\mu_1, \cdots, \mu_k$ be given by (2.9), set $\mu_0 = \frac{\delta^2}{\mu_1}$ and $\mu_{k+1} = 0$. Define the following annulus

$$
A_i := B(0, \sqrt{\mu_i \mu_{i-1}}) \backslash B(0, \sqrt{\mu_i \mu_{i+1}}), \qquad \text{for} \quad i = 1, \cdots, k.
$$

We observe that $B(0,\delta) = \bigcup_{i=1}^k A_i$. On each A_i, the leading term in $\sum_{j=1}^k U_{\mu_j}$ is U_{μ_i}.

Then we have

$$
\begin{aligned}
-(q+1)J_3 \quad = \quad & \lambda \sum_{l=1}^k \int_{A_l} \left[\left(U_{\mu_l} + \sum_{j=1, j\neq l}^k U_{\mu_j} \right)^{q+1} - U_{\mu_l}^{q+1} - (q+1) U_{\mu_l}^q \sum_{j=1, j\neq l}^k U_{\mu_j} \right] \\
& + \lambda \sum_{l=1}^k \int_{A_l} U_{\mu_l}^{q+1} + \lambda (q+1) \sum_{l=1}^k \int_{A_l} \sum_{j=1, j\neq l}^k U_{\mu_l}^q U_{\mu_j} + \lambda \int_{\mathbb{R}^N \backslash B(0,\delta)} \left(\sum_{j=1}^k U_{\mu_j} \right)^{q+1} \\
:= \quad & J_{3,1} + J_{3,2} + J_{3,3} + J_{3,4}.
\end{aligned}
$$

By the mean value theorem, for some $t \in [0,1]$, we have

$$
\begin{aligned}
J_{3,1} \quad = \quad & \lambda \frac{q(q+1)}{2} \sum_{l=1}^k \int_{A_l} \left(U_{\mu_l} + t \sum_{j=1, j\neq l}^k U_{\mu_j} \right)^{q-1} \left(\sum_{j=1, j\neq l}^k U_{\mu_j} \right)^2 \\
\leq \quad & C\lambda \sum_{j,l=1, j\neq l}^k \int_{A_l} w_{\mu_l}^{q-1} w_{\mu_j}^2 + C\lambda \sum_{i,j,l=1,\ i,j\neq l}^k \int_{A_l} w_{\mu_i}^{q-1} w_{\mu_j}^2 .
\end{aligned}
$$

Since

$$
\begin{aligned}
\sum_{j,l=1, j\neq l}^k \int_{A_l} w_{\mu_l}^{q-1} w_{\mu_j}^2 \quad = \quad & \sum_{j,l=1, j\neq l}^k \int_{A_l} (w_{\mu_l}^{q-1} w_{\mu_j}^{\frac{q-1}{q}}) w_{\mu_j}^{\frac{q+1}{q}} \\
\leq \quad & \sum_{j,l=1, j\neq l}^k \left(\int_{A_l} w_{\mu_l}^q w_{\mu_j} \right)^{\frac{q-1}{q}} \left(\int_{A_l} w_{\mu_j}^{q+1} \right)^{\frac{1}{q}}, \qquad (6.6)
\end{aligned}
$$

and

$$(6.7)\quad \sum_{i,j,l=1,\ i,j\neq l}^{k}\int_{A_l} w_{\mu_i}^{q-1}w_{\mu_j}^2 \quad\leq\quad \sum_{i,j,l=1,\ i,j\neq l}^{k}\left(\int_{A_l} w_{\mu_i}^{q+1}\right)^{\frac{q-1}{q+1}}\left(\int_{A_l} w_{\mu_j}^{q+1}\right)^{\frac{2}{q+1}}.$$

If $j>l$, then

$$\begin{aligned}\int\limits_{A_l} w_{\mu_l}^q w_{\mu_j}\,dz &= \alpha_N^{q+1}\int\limits_{\sqrt{\mu_l\mu_{l+1}}\leq|z|\leq\sqrt{\mu_l\mu_{l-1}}}\frac{\mu_l^{\frac{N-2}{2}q}}{(\mu_l^2+|z|^2)^{\frac{N-2}{2}q}}\frac{\mu_j^{\frac{N-2}{2}}}{(\mu_j^2+|z|^2)^{\frac{N-2}{2}}}dz\\ =\quad &\left(\frac{\mu_j}{\mu_l}\right)^{\frac{N-2}{2}}\mu_l^{-\frac{N-2}{2}q+\frac{N+2}{2}}\left[\alpha_N^{q+1}\int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^{\frac{N-2}{2}q}}\frac{1}{|z|^{N-2}}dz+o(1)\right].\end{aligned}\tag{6.8}$$

If $j<l$, then

$$\begin{aligned}\int\limits_{A_l} w_{\mu_l}^q w_{\mu_j}\,dx &= \alpha_N^{q+1}\int\limits_{\sqrt{\mu_l\mu_{l+1}}\leq|z|\leq\sqrt{\mu_l\mu_{l-1}}}\frac{\mu_l^{\frac{N-2}{2}q}}{(\mu_l^2+|z|^2)^{\frac{N-2}{2}q}}\frac{\mu_j^{\frac{N-2}{2}}}{(\mu_j^2+|z|^2)^{\frac{N-2}{2}}}dz\\ =\quad &\left(\frac{\mu_l}{\mu_j}\right)^{\frac{N-2}{2}}\mu_l^{-\frac{N-2}{2}q+\frac{N+2}{2}}\alpha_N^{q+1}\int\limits_{\sqrt{\frac{\mu_{l+1}}{\mu_l}}\leq|z|\leq\sqrt{\frac{\mu_{l-1}}{\mu_l}}}\frac{1}{(1+|z|^2)^{\frac{N-2}{2}q}}\frac{1}{(1+(\frac{\mu_l}{\mu_j})^2|z|^2)^{\frac{N-2}{2}}}dz\\ \leq\quad &\left(\frac{\mu_l}{\mu_j}\right)^{\frac{N-2}{2}}\mu_l^{-\frac{N-2}{2}q+\frac{N+2}{2}}\alpha_N^{q+1}\int\limits_{\sqrt{\frac{\mu_{l+1}}{\mu_l}}\leq|z|\leq\sqrt{\frac{\mu_{l-1}}{\mu_l}}}\frac{1}{(1+|z|^2)^{\frac{N-2}{2}q}}dz.\end{aligned}\tag{6.9}$$

For $i\neq l$, we have

$$(6.10)\quad \int\limits_{A_l} w_{\mu_i}^{q+1}\leq C\mu_i^{-\frac{N-2}{2}q+\frac{N+2}{2}}\begin{cases}(\frac{\mu_l}{\mu_i})^{\frac{N}{2}} & \text{if}\quad i\leq l-1<l;\\ (\frac{\mu_i^2}{\mu_l\mu_{l-1}})^{\frac{N-2}{2}q-1} & \text{if}\quad i\geq l+1>l.\end{cases}$$

From (6.6)-(6.10), (1.4) and (2.9), we get $J_{3,1}=o(\varepsilon)$.
Moreover,

$$\begin{aligned}J_{3,2} &= \lambda\sum_{l=1}^{k}\int_{A_l} w_{\mu_l}^{q+1}+\lambda\sum_{l=1}^{k}\int_{A_l}(U_{\mu_l}^{q+1}-w_{\mu_l}^{q+1})\\ &= \varepsilon\Lambda_1^{\frac{N+2-(N-2)q}{2}}\lambda\int\limits_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^{\frac{(N-2)(q+1)}{2}}}dz+o(\varepsilon).\end{aligned}$$

From (6.8) and (6.9), we have

$$J_{3,3}\leq C\lambda\sum_{l=1}^{k}\int_{A_l}\sum_{j=1,j\neq l}^{k}U_{\mu_l}^q U_{\mu_j}\leq C\lambda\sum_{l=1}^{k}\int_{A_l}\sum_{j=1,j\neq l}^{k}w_{\mu_l}^q w_{\mu_j}=o(\varepsilon).$$

Finally,

$$J_{3,4} \quad = \quad \lambda \int_{\mathbb{R}^N \setminus B(0,\delta)} \left(\sum_{j=1}^{k} U_{\mu_j} \right)^{q+1} \quad \leq C \sum_{j=1}^{k} \int_{\mathbb{R}^N \setminus B(0,\delta)} w_{\mu_j}^{q+1} dz = o(\varepsilon).$$

Thus we get

$$J_3 = -\varepsilon \Lambda_1^{\frac{N+2-(N-2)q}{2}} \frac{\lambda}{q+1} \int_{\mathbb{R}^N} \frac{1}{(1+|z|^2)^{\frac{(N-2)(q+1)}{2}}} dz + o(\varepsilon). \tag{6.11}$$

From (6.3), (6.4), (6.5) and (6.11), we obtain (5.3). □

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W. Chen–Departamento de Ingeniería Matemática and CMM, Universidad de Chile, Casilla 170 Correo 3, Santiago, Chile.
E-mail address: wjchen1102@gmail.com

J. Dávila–Departamento de Ingeniería Matemática and CMM, Universidad de Chile, Casilla 170 Correo 3, Santiago, Chile.
E-mail address: jdavila@dim.uchile.cl

I. Guerra–Departamento de Matemática y Ciencia de la Computacion, Universidad de Santiago de Chile, 9170125, Santiago, Chile.
E-mail address: ignacio.guerra@usach.cl